UNIVERSITY "Sts. CYRIL AND METHODIUS"

FACULTY OF MECHANICAL ENGINEERING

TEMPUS JEP DEREC

COURSE:

FLUID MECHANICS

Lecture notes prepared by

Prof. Dr. Aleksandar Nošpal

SKOPJE
2008

Contents

University Ss Cyril and Methodius

TEMPUS JEP DEREC

Course syllabus prepared by *Prof. Aleksandar Nošpal* from *the University Ss Cyril and Methodius* and *Prof. Petros Anagnostoupolos* from *the Aristotle University of Thessaloniki*

COURSE TITLE: ***FLUID MECHANICS***

COURSE NUMBER:

Time schedule:

9 credits (25x9=225 learning hours)

ECTS distribution:
Lecture time: 89 hours (54 hours lectures + 35 hours tutorials)
Laboratory work: 10 hours
Self study: 120 hours
Testing, exams, presentations: 6 hours
TOTAL: 225 hours

Week class distribution (lecturs + tutorials/lab. practising): 4+5

COURSE CONVENOR *Prof. Dr. Aleksandar Nošpal*

COURSE AIMS:

Knowledge of:

fundamentals and application of the Fluid Mechanics; basic laws and fundamental concepts of fluid flows; basic considerations of Experimental Fluid Mechanics and CFD; methods and examples of Applied Fluid Mechanics - characteristic for the engineering practice and especially Environmental and Resources Engineering.

LEARNING OUTCOMES:

By the end of this module students should be able to:

solve basic and practical fluid flow problems from the field of Applied Fluid Mechanics; be better prepared for further knowledge acceptance needed for experimental and CFD methods; understand better other subjects in the area of Environmental and Resources Engineering.

TEACHING AND LEARNING METHODS

lecturing, tutorials, laboratory work, presentation of video materials, use of Internet, self-study, homework preparation

DETAILS OF ASSESSMENT INSTRUMENTS

active participation on classes, homework and lab assignments, knowledge assessment on tests

SUMMARY DESCRIPTION OF ASSESSMENT

Grading system is given by the following table:

Assessment	Points	Percentage
Active participation	30	10
Homeworks 1,2,3	30	10
Homeworks 4,5,6	30	10
Midterm test	90	30
Final test	99	33
Lab work	21	7
Total	300	100

Quality grading is realized by the following table:

Points	Grade	Equivalent
271-300	10	A
241-270	9	B
211-240	8	C
181-210	7	D
151-180	6	D-

BACKGROUND READING - BASIC TEXTS

Street R.L., Watters G.Z., Vennard J.K., *Elementary Fluid Mechanics*, John Wiley & Sons, 7th editiond, 1996, ISBN: 978-0-471-01310-5

Bundalevski T., : *Mechanics of Fluids* (in Macedonian), University Ss Cyril and Methodius, publisher MB-3, Skopje, 1995, ISBN 9989-704-01-5

Virag Z., : *Fluid Mechanics - selected chapters, examples and problems* (in Croation), University of Zagreb, Faculty of mechanical Engineering, 2002

Nospal A.: "*Fluid Flow Measurments and Instrumentation*" (in Macedonian), University Ss Cyril and Methodius, publisher MB-3, Skopje, 1995, iSBN 9989-704-02-3

Stojkovski V., Nošpal A., Kostic.Z.,: "*Practicum for Laboratory Works for the Subject Fluid Flow Measurements and Instrumentation*", edition for students of the Faculty of Mechanical Engineering, Skopje, 1993.

Nošpal A., Stojkovski V.,: *Fluid Mechanics - Prepared lecures and tutorials material for the DEREC subject*, edition of the Faculty of Mechanical Engineering, Skopje, 2007/2008

Professors from EU DEREC Universities: *Fluid Mechanics - Prepared lecures and tutorials material for the DEREC subject*, Educational Material prepared by professors from EU DEREC Universities, 2007/2008

SYLLABUS:

1. Introduction to the Fluid Mechanics
   - 1.1. The importance of the Fluid Mechanics for the Science and Engineering; the importance for Environmental and Resources Engineering
   - 1.2. Fundamental dimensions, dimensional homogeneity and fundamental units of measurement.
   - 1.3. Properties and states of fluids - pressure, temperature, density, specific weight, viscosity, specific heat, internal energy, bulk modulus of elasticity and compressibility, velocity of sound, equations of state
   - 1.4. Forces on a fluid element and pressure
2. Statics of fluids
   - 2.1. Basic laws - hydrostatic pressure, Euler's equations of equilibrium
   - 2.2. Equilibrium in gravity field - incompressible fluid in gravity field, hydrostatic manometers, Pascal's law, relative equilibrium of fluid (translation and rotation of a liquid container), pressure force on a flat and curved surface, buoyant forces
3. Kinematics of flow
   - 3.1. Flow field - properties of flow field, Lagrangean versus Eulerian aproach, steady and unsteady flow
   - 3.2. Velocity, stream line versus path line, stream function, stream tube, velocity gradient and shear
   - 3.3. Volume flow, flux and circulation
   - 3.4. Continuity equations
   - 3.5. Acceleration
4. Dynamics of inviscid (ideal) fluid flow
   - 4.1. Forces on a inviscid fluid element, 3-D and 2-D flows, Euler's equations for inviscid fluid flow
   - 4.2. One dimensional gravity flow - Bernoulli's equation
   - 4.3. Potential flow - differential equations, Cauchy-Lagrange and Bernoulli equation
   - 4.4. The continuity equation in integral form
   - 4.5. Equations of momentum and energy
5. Some elementary flows of inviscid fluid
   - 5.1. Stream tube control volume. Basic equations for flows through a stream tube
   - 5.2. Some examples of steady flow of incompressible fluid - Venturi tube, discharge through nozzles from a reservoir into the atmosphere, submerged discharge, flow through a rotating tube, cavitation
   - 5.3. Basic consideration of compressible fluid flow
   - 5.4. Some examples for the momentum equations application - force on a bended pipe, jet reaction, basic equation of the turbo-machines
6. Some fundamental concepts of viscous fluid flow
   - 6.1. General concept of viscous fluid flow - Newton's law for shear stress, flow classification, laminar versus turbulent flow
   - 6.2. Fundamental equations for laminar flow - stresses in a viscous fluid flow, friction forces, Navier-Stokes equations
   - 6.3. Fundamental concepts and solutions of the governing equations for some cases of laminar flow
   - 6.4. Fundamental concepts and equations for creeping motions and two-dimensional boundary layer
   - 6.5. The notion of resistance, drag, and lift
   - 6.6. Basic concepts of incompressible viscous fluid turbulent flow - Reynolds experiment and Reynolds number, velocity in turbulent flow, Reynolds equations for turbulent flow of incompressible fluid
   - 6.7. Concepts for solving governing equations of viscous fluid flow - features of theoretical methods, experimental and semi-empirical approach, CFD approach.
7. Basic consideration of Experimental Fluid Mechanics
   - 7.1. Basic approach to the Dimensional Analysis - dimensional homogeneity, Rayleigh method, the significance of non-dimensional relationships and numbers, Vaschy's theorem.
   - 7.2. Basic approach to the experimental investigation and application of the similarity theory - similarity criteria for characteristic flow conditions

8. Methods and examples of Applied Fluid Mechanics

    8.1. Basic equations of flow in conduits and pipes - velocity distribution and average velocity; pressure; continuity equation; Bernoulli equation; momentum law; energy losses linear and local losses
    8.2. Laminar and turbulent incompressible flows in pipes - velocity profiles for turbulent flow, velocity and friction laws, roughness effects, examples for pipe-flow computation
    8.3. Incompressible flow in noncircular ducts - friction losses in closed conduits, two dimensional flows
    8.4. Flow in prismatic open channels - one dimensional open-channel equations, head-loss equations, velocity and friction laws for two-dimensional channels, computation examples
    8.5. Immersed bodies, drag and lift - hydrodynamic forces and force coefficients, drag of symmetrical bodies, lift and drag of nonsymmetrical bodies
    8.6. Basic approach to turbulent jets and diffusion processes - free turbulence, diffusion processes in nonhomogeneous fluids
    8.7. Basic approach to multiphase flow

Distribution of the material by weeks:

Week 1:	1.1.; 1.2.;
Week 2:	1.3.;
Week 3:	1.4.; 2.1
Week 4:	2.2
Week 5:	3.1.; 3.2.; 3.3;
Week 6:	3.4.; 4.1.; 4.2.;
Week 7:	4.3; 4.4; 4.5.
Week 8:	5.1; 5.2.
Week 9:	5.3; 5.4.
Week 10:	6.1.; 6.2.; 6.3.; 6.4.
Week 11:	6.5.; 6.6.; 6.7.
Week 12:	7.1.; 7.2.
Week 13:	8.1.; 8.2.;
Week 14:	8.3.; 8.4.; 8.5.;.
Week 15:	8.6.; 8.7.;

UNIT GUIDE **TEMPUS JEP DEREC**

Unit Title:	***FLUID MECHANICS***
Mode:	
Co Requisites:	*Solid Mechanics*
Pre Requisites	*Mathematics II and Solid Mechanics*
Lectures:	54 hours
Tutorials:	35 hours
Lab practicing:	10 hours
Individual Study Hours:	120 hours
Study Hours:	225 hours
Method of Assessment:	active participation on classes, homework and lab assignments, knowledge assessment on tests
Study year:	II
Semester:	III
ECST Credit Value:	9
Web support	http://www.derec.ukim.edu.mk
Module:	
Level:	undergraduate
Subject Area:	Environmental and Resources Engineering
Unit Coordinator:	Prof. Dr Aleksandar Nošpal
Version:	english/macedonian

MOTIVATION

To ensure knowledge transfer to the students in a field and subject very important for the foreseen studies of Environmental and Resources Engineering.

SHORT DESCRIPTION

The unit (subject) is planned acoording the following main course parts:
Introduction to the Fluid Mechanics; Statics of Fluids; Kinematics of Fluids; Dynamics of ideal fluid flow; Some elementary flows of inviscid fluid; Some fudamental concepts of viscous fluid flow; Basic consideration of Experimental Fluid Mechanics; Method and Examples of Applied Fluid Mechanics

AIMS

Knowledge of:

fundamentals and application of the Fluid Mechanics; basic laws and fundamental concepts of fluid flows; basic considerations of Experimental Fluid Mechanics and CFD; methods and examples of Applied Fluid Mechanics - characteristic for the engineering practice and especially Environmental and Resources Engineering.

LEARNING OUTCOMES

Students who complete this course should be able to perform the following tasks:

to solve basic and practical fluid flow problems from the field of Applied Fluid Mechanics; to be better prepared for further knowledge acceptance needed for experimental and CFD methods; to understand better other subjects in the area of Environmental and Resources Engineering.

TRANSFERABLE SKILLS

At the end of the unit students will be able to:
continue more efficiently his further studies in the field of Environmental and Resources Engineering, or other engineering studies if he plans such a transfer.

INDICATIVE CONTENT

Introduction to the Fluid Mechanics: The importance of the Fluid Mechanics; Fundamental dimensions and units of measurement; Properties and states of fluids; Forces on a fluid element and pressure.

Statics of fluids: Basic laws - hydrostatic pressure, Euler's equations; Equilibrium in gravity field -incompressible fluid in gravity field, hydrostatic manometers, Pascal's law, relative equilibrium of fluid, pressure force on a flat and curved surface, buoyant forces.

Kinematics of flow: Flow field - properties of flow field, Lagrangean versus Eulerian aproach, steady and unsteady flow; Velocity, stream line and stream function, stream tube, velocity gradient and shear; Volume flow, flux and circulation; Continuity equations; Acceleration.

Dynamics of inviscid fluid flow: Forces on a inviscid fluid element, Euler's equations for inviscid fluid flow;One dimensional gravity flow - Bernoulli's equation; Potential flow - Cauchy-Lagrange and Bernoulli equation; The continuity equation in integral form; Equations of momentum and energy.

Some elementary flows of inviscid fluid: Stream tube control volume. Basic equations for flows through a stream tube; Some examples of steady flow of incompressible fluid - Venturi tube, discharge through nozzles from a reservoir into the atmosphere, submerged discharge, flow through a rotating tube, cavitation; Basic consideration of compressible fluid flow; Some examples for the momentum equations application - force on a bended pipe, jet reaction, basic equation of the turbo-machines.

Some fundamental concepts of viscous fluid flow: General concept of viscous fluid flow - Newton's law for shear stress, flow classification; Fundamental equations for laminar flow - Navier-Stokes equations; Bases of creeping motions and two-dimensional boundary layer; The notion of resistance, drag, and lift; Basic concepts of incompressible viscous fluid turbulent flow - Reynolds number, velocity in turbulent flow, Reynolds equations; Concepts for solving governing equations - experimental and CFD approach.

Basic consideration of Experimental Fluid Mechanics: Basic approach to the Dimensional Analysis - Rayleigh's method and Vaschy's theorem; Basic approach to the experimental investigation and application of the similarity theory - similarity criteria for characteristic flow conditions.

Methods and examples of Applied Fluid Mechanics: Basic equations of flow in conduits and pipes - velocity distribution, pressure, continuity equation, Bernoulli equation, momentum law, energy losses; Laminar and turbulent incompressible flows in pipes - velocity profiles, velocity and friction laws, roughness effects, examples for pipe-flow computation; Bases of incompressible flow in noncircular ducts; Bases of flow in prismatic open channels; Immersed bodies, drag and lift - hydrodynamic forces and force coefficients, drag and lift; Basic approach to turbulent jets and diffusion processes; Basic approach to multiphase flow.

CONTENT

WEEKLY TEACHING PLAN AND LEARNING PROGRAMME

Week	Lectures	Tutorials and Lab practicing
1	1. Introduction to the Fluid Mechanics: The importance of the Fluid Mechanics for the Science and Engineering; the importance for Environmental and Resources Engineering; Fundamental dimensions, dimensional homogeneity and fundamental units of measurement;	Video materials presentations of H. Rouse (from IIHR) and A. Shapiro (from MIT). Notion of some useful web sites.
2	Properties and states of fluids - pressure, temperature, density, specific weight, viscosity, specific heat, internal energy, bulk modulus of elasticity and compressibility, velocity of sound, equations of state;	Examples and problems from measurment units. Examples and problems from fluids properties.
3	Forces on a fluid element and pressure. 2. Statics of fluids: Basic laws - hydrostatic pressure, Euler's equations of equilibrium	Examples and problems from Statics of fluids.
4	Equilibrium in gravity field - incompressible fluid in gravity field, hydrostatic manometers, Pascal's law, relative equilibrium of fluid (translation and rotation of a liquid container), pressure force on a flat and curved surface, buoyant forces	Examples and problems from Statics of fluids.
5	3. Kinematics of flow: Flow field - properties of flow field, Lagrangean versus Eulerian aproach, steady and unsteady flow; Velocity, stream line versus path line, stream function, stream tube, velocity gradient and shear; Volume flow, flux and circulation;	Lab measurements of some fluid properties. Lab measurements of pressure.
6	Continuity equations; Acceleration 4. Dynamics of inviscid (ideal) fluid flow: Forces on a inviscid fluid element, 3-D and 2-D flows, Euler's equations for inviscid fluid flow; One dimensional gravity flow - Bernoulli's equation;	Examples and problems from Dinamics of inviscid fluid flow.

7	Potential flow - differential equations, Cauchy-Lagrange and Bernoulli equation; The continuity equation in integral form; Equations of momentum and energy.	Examples and problems from Dinamics of inviscid fluid flow.
8	5. Some elementary flows of inviscid fluid Stream tube control volume. Basic equations for flows through a stream tube; Some examples of steady flow of incompressible fluid - Venturi tube, discharge through nozzles from a reservoir into the atmosphere, submerged discharge, flow through a rotating tube, cavitation;	Examples and problems from some elementary flows of inviscid fluid.
9	Basic consideration of compressible fluid flow; Some examples for the momentum equations application - force on a bended pipe, jet reaction, basic equation of the turbo-machines.	Problems for the momentum equations application.
10	6. Some fundamental concepts of viscous fluid flow General concept of viscous fluid flow - Newton's law for shear stress, flow classification, laminar versus turbulent flow; Fundamental equations for laminar flow - stresses in a viscous fluid flow, friction forces, Navier-Stokes equations; Fundamental concepts and equations for creeping motions and two-dimensional boundary layer	Some basic laboratory measurments of fluid flow velocity; and volume and mass flow rate.
11	The notion of resistance, drag, and lift; Basic concepts of incompressible viscous fluid turbulent flow - Reynolds experiment and Reynolds number, velocity in turbulent flow, Reynolds equations for turbulent flow of incompressible fluid; Concepts for solving governing equations of viscous fluid flow - experimental and semi-empirical approach, CFD approach.	Some video presentations for the fundamental concepts of viscous fluid flow. Some examples for solving the governing equations - experimental and CFD aproach.
12	7. Basic consideration of Experimental Fluid Mechanics Basic approach to the Dimensional Analysis - dimensional homogeneity, Rayleigh method, the significance of non-dimensional relationships and numbers, Vaschy's theorem; Basic approach to the experimental investigation and application of the similarity theory - similarity criteria for characteristic flow conditions.	Some examples and problems for Dimensional Analysis Application. Some examples and problems for Similarity Theory Application.
13	8. Methods and examples of Applied Fluid Mechanics Basic equations of flow in conduits and pipes - velocity distribution and average velocity, pressure, continuity equation, Bernoulli equation, momentum law, energy losses linear and local losses; Laminar and turbulent incompressible flows in pipes - velocity profiles for turbulent flow, velocity and friction laws, roughness effects, examples for pipe-flow computation;	Examples and problems from Appled Fluid Mechanics.
14	Incompressible flow in noncircular ducts - friction losses in closed conduits, two dimensional flows; Flow in prismatic open channels - one dimensional open-channel equations, head-loss equations, velocity and friction laws for two-dimensional channels, computation examples; Immersed bodies, drag and lift - hydrodynamic forces and force coefficients, drag of symmetrical bodies, lift and drag of nonsymmetrical bodies;	Examples and problems from Applied Fluid Mechanics. Examples for some measurements in Applied Fluid Mechanics.
15	Basic approach to turbulent jets and diffusion processes - free turbulence, diffusion processes in nonhomogeneous fluids; Basic approach to multiphase flow.	Examples and problems from Appled Fluid Mechanics.

TEACHING METHOD

lecturing, tutorials, laboratory work, presentation of video materials, use of Internet, self-study, homework preparation

ASSESSMENT METHOD

Active participation on classes - 30 points (10%)

Homework assignments (6 homeworks) - 60 points (20%)

Laboratory work - 21 points (7%)

Knowledge assessment on tests – 189 points (63%)

GRADING

Grading system is given by the following table:

Assessment	Points	Percentage
Active participation	30	10
Homeworks 1,2,3	30	10
Homeworks 4,5,6	30	10
Midterm test	90	30
Final test	99	33
Lab work	21	7
Total	300	100

Quality grading is realized by the following table:

Points	Grade	Equivalent
271-300	10	A
241-270	9	B
211-240	8	C
181-210	7	D
151-180	6	D-

COURSE LEARNING MATERIALS

Textbook

Street R.L., Watters G.Z., Vennard J.K., *Elementary Fluid Mechanics*, John Wiley & Sons, 7th editiond, 1996, ISBN: 978-0-471-01310-5

Bundalevski T., : *Mechanics of Fluids* (in Macedonian), University Ss Cyril and Methodius, publisher MB-3, Skopje, 1995, ISBN 9989-704-01-5

Nošpal A., Stojkovski V.,: *Fluid Mechanics - Prepared lecures and tutorials material for the DEREC subject*, edition of the Faculty of Mechanical Engineering, Skopje, 2007/2008

Professors from EU DEREC Universities: *Fluid Mechanics - Prepared lecures and tutorials material for the DEREC subject*, Educational Material prepared by professors from EU DEREC Universities, 2007/2008

Tutorial

Nošpal A., Stojkovski V.,: *Fluid Mechanics - Prepared lecures and tutorials material for the DEREC subject*, edition of the Faculty of Mechanical Engineering, Skopje, 2007/2008

Professors from EU DEREC Universities: *Fluid Mechanics - Prepared lecures and tutorials material for the DEREC subject*, Educational Material prepared by professors from EU DEREC Universities, 2007/2008

Lab practicum

Nospal A.: "*Fluid Flow Measurments and Instrumentation*" (in Macedonian), University Ss Cyril and Methodius, publisher MB-3, Skopje, 1995, iSBN 9989-704-02-3

Stojkovski V., Nošpal A., Kostic.Z.,: "*Practicum for Laboratory Works for the Subject Fluid Flow Measurements and Instrumentation*", edition for students of the Faculty of Mechanical Engineering, Skopje, 1993.

Stojkovski V., Nošpal A.,: *Fluid Mechanics - Prepared lecures and tutorials material for the DEREC subject*, edition of the Faculty of Mechanical Engineering, Skopje, 2007/2008

Professors from EU DEREC Universities: *Fluid Mechanics - Prepared lecures and tutorials material for the DEREC subject*, Educational Material prepared by professors from EU DEREC Universities, 2007/2008

Web support

http://www.derec.ukim.edu.mk

BACKGROUND

TEMPUS JEP DEREC MATERIALS

UNIVERSITIES CONSORTIUM: *University of Florence, University Sts. Cyril and Methodius, Aristotele University of Thessaloniki, Ruhr University Bochum, Vienna University of Technology*

1. Introduction to the Fluid Mechanics

1.1. The importance of the Fluid Mechanics for the Science and Engineering; the importance for Environmental and Resources Engineering

Definition:

physical science dealing with the action of fluids at rest or in motion, and with *engineering* applications and devices using fluids.

A *fluid* is defined as a substance that continually deforms (flows) under an applied shear stress regardless of the magnitude of the applied stress. It is a subset of the phases of matter and includes *liquids*, *gases*, *plasmas* and, to some extent, *plastic solids*.

Fluids are also divided into *liquids* (*incompressible fluids*) and *gases* (*compressible fluids*).

Physics ⇒ *Mechanics* ⇒ *Fluid Mechanics*

Continuum Mechanics

Fluid mechanics is basic to such diverse fields as *aeronautics* , *chemical*, *civil*, *mechanical engineering*, *meteorology*, *naval architecture*, *oceanography*.

Fluid mechanics can be subdivided into two major areas:

fluid statics, or *hydrostatics*, which deals with fluids at rest, and
fluid dynamics, concerned with fluids in motion.

The term *hydrodynamics* is applied to the flow of liquids or to low-velocity gas flows in which the gas can be considered as being essentially incompressible.

Aerodynamics or *gas dynamics* is concerned with the behaviour of gases when velocity and pressure changes are sufficiently large to require inclusion of the compressibility effects.

Hydraulics:
application of *fluid mechanics* to *engineering devices* involving *liquids*, usually water or oil.

Hydraulics deals with such problems as the *flow of fluids through pipes* or in *open channels* and the *design of storage dams*, *pumps*, and *water turbines*. With other devices it deals with the *control or use of liquids*, such as nozzles, valves, jets, and flowmeters.

Applications of fluid mechanics include also *jet propulsion*, *gas and vapor turbines*, *compressors* etc.

∴ *Fluid Mechanics - extremly important for Environmental and Resources engineering.*

Web sites references:

http://en.wikipedia.org/wiki/Fluid_mechanics; uk.encarta.msn.com/encyclopedia_761578780/Fluid_Mechanics.html

www.britannica.com/eb/article-9110311/fluid-mechanics

http://ocw.mit.edu/OcwWeb/index.htm; http://www.iihr.uiowa.edu; The Science of All Things Fluid

⇒ Video Presentation:

Hunter Rouse: *Introduction to the Study of Fluid Motion*

1.2. Fundamental dimensions, dimensional homogeneity and fundamental units of measurement

Equations in physics have dimensional homogeneity - not only because of their theoretical derivation but also due to the way of measurements of the physical quantities.

Definition:

All members in an equation have the same physical meaning and are expressed with same measurement units.

Example:

A form of the Bernoulli equation

$$p + \gamma h + \frac{\rho v^2}{2} = p_0 + \gamma h_0 + \frac{\rho v_0^2}{2}$$

All three members are/present pressure:

p - flow pressure;

γh - hydrostatic pressure;

$\frac{\rho v^2}{2}$ - dynamic pressure

All members have same dimensional formula - $[FL^{-2}]$ i.e. $[ML^{-1}T^{-2}]$,
and are expressed with same units - $[N/m^2]$.

⇒

Fundamental Quantities, Dimensions and Units:

⇒ *Fundamental Quantities* in Mechanics and Fluid Mechanics:

- *Length, Mass, Time, Temperature*
  - ⇒ *Fundamental Dimensions* - L,M,T,θ
  - ⇒ *Fundamental Units of Measurement (SI)* - m, kg, s, K
- *Length, Force, Time, Temperature*
  - ⇒ *Fundamental Dimensions* - L,F,T,θ
  - ⇒ *Fundamental Units of Measurement (SI)* - m, N, s, K

⇒ *Dimensional formulae*

Dimensional Formulae and Measurment Units

a) Geometric Quantity

Quantity	*Symbol*	*Dimensions*		*Measurement units*	
		M,L,T,θ	F,L,T,θ	SI	Old technical
Length	*l, r, a, b*	L	L	m	m
Area/Surface	*A*	L^2	L^2	m^2	m^2
Volume	*V*	L^3	L^3	m^3	m^3
Curvature	*C=1/R*	L^{-1}	L^{-1}	m^{-1}	m^{-1}
Hydraulic radius	*R*	L	L	m	m
Roughness	*k*	L	L	m	m
Wave length	λ	L	L	m	m
Angle	$\alpha, \beta, \gamma,...$	–	–	rad ; ^{0}	rad ; ^{0}
Resistance moment /First moment of area	*W*	L^3	L^3	m^3	m^3
Geometric moment of inertia	*I*	L^4	L^4	m^4	m^4

b) Kinematic Quantities

Quantity	*Symbol*	*Dimensions*		*Measurement units*	
		M,L,T,θ	F,L,T,θ	SI	Old technical
Time	t	T	T	s	s
Rate of deformation	$\partial v_j / \partial x_i$	T^{-1}	T^{-1}	s^{-1}	s^{-1}
Angular velocity	ω	T^{-1}	T^{-1}	s^{-1}	s^{-1}
Frequency	f	T^{-1}	T^{-1}	s^{-1}	s^{-1}
Angular acceleration	$\dot{\omega}$	T^{-2}	T^{-2}	s^{-2}	s^{-2}
Velocity	u, v, w	LT^{-1}	LT^{-1}	m/s	m/s
Acceleration	$a, \dot{v}$	LT^{-2}	LT^{-2}	m/s^2	m/s^2
Volume flow rate	Q	L^3T^{-1}	L^3T^{-1}	m^3/s	m^3/s
2D flow rate	q	L^2T^{-1}	L^2T^{-1}	m^2/s	m^2/s
Circulation	Γ	L^2T^{-1}	L^2T^{-1}	m^2/s	m^2/s
Kinematic viscosity	ν	L^2T^{-1}	L^2T^{-1}	m^2/s	m^2/s
Vorticity	$\vec{\Omega} = \vec{\nabla} \times \vec{v}$	T^{-1}	T^{-1}	s^{-1}	s^{-1}

c) Dynamic Quantities

Quantity	*Symbol*	*Dimensions*		*Measurement units*	
		M,L,T,θ	F,L,T,θ	SI	Old technical
Mass	m	M	FT^2L^{-1}	kg	kps^2/m
Force	F	MLT^{-2}	F	kgm/s^2=N	kp
Pressure	p	$ML^{-1}T^{-2}$	FL^{-2}	N/m^2	kp/m^2
Stress	σ, τ	$ML^{-1}T^{-2}$	FL^{-2}	N/m^2	kp/m^2
Pressure gradient	$\Delta p/\Delta x_j$	$ML^{-2}T^{-2}$	FL^{-3}	N/m^3	kp/m^3
Density	ρ	ML^{-3}	FT^2L^{-4}	kg/m^3	kps^2/m^4
Specific weight	γ	$ML^{-2}T^{-2}$	FL^{-3}	N/m^3	kp/m^3
Momentum, Impulse	$\vec{M}$, $\vec{K} = m\vec{v}$	MLT^{-1}	FT	kgm/s	kps
Angular momentum	$H=mr^2\omega$	ML^2T^{-1}	FLT	kgm^2/s	kpms
Momentum flux, Momentum force	$\vec{F}_R = \rho Q\vec{v}$	MLT^{-2}	F	kgm/s^2	kp
Moment of momentum	$\vec{M}_k, \vec{M}_m$	ML^2T^{-1}	FLT	Nms	kpms
Moment of force, Torque	$\vec{M}, \vec{M}_T, T$	ML^2T^{-2}	FL	Nm	kpm
Mass moment of inertia	J	ML^2	FLT^2	kgm^2	$kpms^2$
Relative atomic mass	A	1	1	1	1
Relative molecular mass	M	1	1	1	1
Energy	E	ML^2T^{-2}	FL	kgm^2/s^2=J	kpm
Work	W	ML^2T^{-2}	FL	J=Nm	kpm
Hydraulic head	$h = v^2/2g+p/\gamma+z$	L	L	Nm/N	kpm/kp
Energy per unit mass	$gh = v^2/2+p/\rho+gz$	L^2T^{-2}	L^2T^{-2}	m^2/s^2	m^2/s^2
Power	P	ML^2T^{-3}	FLT^{-1}	W=J/s	kpm/s
Dynamic viscosity	μ, η	$ML^{-1}T^{-1}$	FTL^{-2}	Ns/m^2	kps/m^2
Eddy viscosity	ε	$ML^{-1}T^{-1}$	FTL^{-2}	kg/ms	kps/m^2
Modulus of elasticity	E	$ML^{-1}T^{-2}$	FL^{-2}	N/m^2	kp/m^2
Bulk modulus of elasticity	$E_V = \Delta p/(\Delta V/V)$	$ML^{-1}T^{-2}$	FL^{-2}	N/m^2	kp/m^2
Mass flow rate	$\dot{m}, q$	MT^{-1}	FTL^{-1}	kg/s	kps/m
Surface tension	σ	MT^{-2}	FL^{-1}	N/m	kp/m
Mass diffusion coefficient.	k, D	L^2T^{-1}	L^2T^{-1}	m^2/s	m^2/s
Concentration of mass	c	ML^{-3}	FL^4T^{-2}	kg/m^3	kpm^4/s^2

d) Thermodynamic Quantities

Quantity	Symbol	Dimensions		Measurement units	
		M,L,T,θ	F,L,T,θ	SI	Old technical
Temperature	T	θ	θ	K	^{0}C, ^{0}K
Temperature gradient	$\Delta T/\Delta x_i$	$L^{-1}\theta$	$L^{-1}\theta$	K/m	^{0}C/m
Quantity of heat	Q	ML^2T^{-2}	FL	J=Nm	kcal= 427 kpm
Thermal conductivity coefficient	λ	$MLT^{-3}\theta^{-1}$	$FT^{-1}\theta^{-1}$	W/mK	kcal/sm^{0}K
Entropy	S, dS	$ML^2T^{-2}\theta^{-1}$	$FL\theta^{-1}$	J/K	kcal/^{0}K
Enthalpy	I, H	ML^2T^{-2}	FL	J	kcal
Gas constant	R	$L^2T^{-2}\theta^{-1}$	$L^2T^{-2}\theta^{-1}$	J/kgK	kcal/kg^{0}K
Specific heat	c_p, c_v	$L^2T^{-2}\theta^{-1}$	$L^2T^{-2}\theta^{-1}$	J/kgK	kcal/kg^{0}K
Specific entalpy	i, h	L^2T^{-2}	L^2T^{-2}	J/kg	kcal/kg
Specific entropy	s	$L^2T^{-2}\theta^{-1}$	$L^2T^{-2}\theta^{-1}$	J/kgK	kcal/kg^{0}K
Heat flux density	q_H	MT^{-3}	$FL^{-1}T^{-1}$	W/m^2	kcal/m^{2}s
Thermal diffusion coefficient	χ	L^2T^{-1}	L^2T^{-1}	m^2/s	m^2/s

1.3. Properties and states of fluids

- pressure, temperature, density, specific weight, viscosity, specific heat, internal energy, bulk modulus of elasticity and compressibility, velocity of sound, equations of state

Pressure:

Property defined as force per unit area:

$$p = \frac{F_p}{A} \tag{1-1}$$

F_p - force applied on a surface A in a direction perpendicular to that surface.

.

Dimensional formula: $[P] = ML^{-1}T^{-2} = FL^{-2}$.

Units:

International System of Units (SI):

1 Pa = 1 N/m^2 ; 1 bar = 10^5 Pa .

N = kgm/s^2

Old "technical":

1 kp/m^2 ≈ 1 mmH_2O = 9,81 Pa
1 at = 1 kp/cm^2 = 0,981 bar
1 atm = 760 mmHg = 1,01325 bar
1 Torr = 1 mmHg = 133,322 Pa

Old British:

1 pound/sq.in. (p.s.i) = 703,1 kp/m^2 = 6,895 kN/m^2
1 pound/sq.ft. (p.s.ft.) = 4,882 kp/m^2 = 47,88 N/m^2

See *Table of units* in the literature

Kinds of pressure - explained later on:

Fluid fow pressure = static presure,
Dynamic pressure,
Total pressure,
Absolute pressure,
Atmospheric pressure,
Gauge pressure,
Vacuum,
Hydrostatic pressure etc

Temperature:

Temperature is a fundamental physical property (quantity) of a system that underlies the common notions of hot and cold - *the level of heat of a fluid*.
On the molecular level, temperature is the result of the motion of particles which make up a substance.
Changes in temperature causes changes in other properties.

Symbol: $T,\ t$

Dimensional formula: θ

Units:

SI:

K = Kelvin

^{0}C = Degree Celsius

$$K = 273{,}16 + [^0C] \qquad (1\text{-}2)$$

Old British:

^{0}F - Degree Fahrenheit's:

$$\left[{}^0C\right] = \frac{5}{9}\left\{\left[{}^0F\right] - 32\right\}; \qquad \left[{}^0F\right] = \frac{9}{5}\left[{}^0C\right] + 32 \qquad (1\text{-}3)$$

Kinds of temperature - explained later on: *Fluid flow temperature*; *Total (stagnation) temperature etc* - simlaer to pressure.

Density:

Density (or specific mass) is defined as ratio of mass and volume:

$$\rho = \frac{\Delta m}{\Delta V} = \frac{m}{V} \tag{1-4}$$

Dimensional formula: ML^{-3}

Units:
SI System: kg/m^3

∴ $\rho = f(p, T)$

For *liquids* (or *incompressible fluids*):

Coefficient of thermal expansion: $\alpha = \frac{1}{V}\frac{dV}{dT} = -\frac{1}{\rho}\frac{d\rho}{dT}$ $1/^0C$

∴ *For values of ρ and α, for diferent fluids, see the corresponding tables in the literature*

In the book of T. Bundalevski the symbol β is used ($\alpha = \beta$)

For *gasses* (or c*ompressible fluids*) - depending the process of change:

- *Equation of state for ideal gass*: $\frac{p}{\rho} = RT$ *R - gass constant* ⇒ see tables in the literature.

- Equation for *Isentropic adiabatic process*: $\frac{p}{\rho^\kappa} = const$

Specific weight:

Defined as weight per unit volume:

$$\gamma = \frac{\Delta G}{\Delta V} = \frac{\Delta mg}{\Delta V} = \rho g \tag{1-5}$$

Dimensional formula: $FL^{-3} = ML^{-2}T^{-2}$

Units:
SI System: N/m^3

∴ *For values for diferent fluids see the corresponding tables in the literature*

Viscosity:

Viscosity is a measure of the resistance of a fluid to deform under shear stress. It is commonly perceived as "thickness", or resistance to flow. ⇒ see *Fig. 1.1*.

Viscosity describes a fluid's internal resistance to flow and may be thought of as a measure of fluid friction.

In general, in any flow, layers move at different velocities and *the fluid's viscosity arises from the shear stress between the layers t*hat ultimately opposes any applied force.

According Isaac Newton (for so called *Newtonian fluids*):

$$\tau = \mu \frac{dv}{dn} \tag{1-6}$$

τ - *shear stress* in N/m^2;

dv/dn - *rate of angular deformation* (velocity gradient) in s^{-1};

μ - *Dynamic (absolute) viscosity* in Ns/m^2; in some literature the symbol η is used ($\mu = \eta$).
SI units: kg/ms=Ns/m^2; P = dyns/cm^2 = g/cms = 0,1 Ns/m^2; cP = 10^{-2} P

Kinematic viscosity, ν, is very often used in the hydraulic computations. Kinematic viscosity is defined as ratio of the dynamic viscosity μ and density ρ:

$$\nu = \frac{\mu}{\rho} \tag{1-7}$$

SI units: m^2/s; St = cm^2/s = 10^{-4} m^2/s ; cSt = 10^{-2} St = 10^{-6} m^2/s ; mSt = 10^{-3} St

$\nu = f(p,T)$

∴ For values of μ and ν, for different fluids, see the corresponding tables and diagrams in the literature

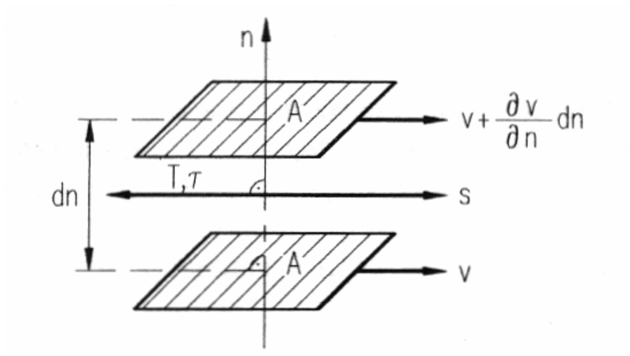


Fig. 1.1: Laminar shear of fluid between two plates

Specific heat, c:

Specific heat capacity, also known simply as *specific heat*, is the ratio of the quantity of heat flowing into a substance per unit mass to the change in temperatutre
= measure of the heat energy required to increase the temperature of one kg of a substance by one Kelvin.

Dimensional formula: $L^2T^{-2}\theta^{-1}$

SI units: J/kgK

c_p = specific heat at constant pressure;

c_v = specific heat at constant volume.

∴ *For values for diferent fluids see the corresponding tables in the literature*

Specific internal energy, u:

Defined as energy per unit mass, due to the kinetic and potential energies bound into the substance by its molecular activity and depends primarily on temperature.

∴ *For values for diferent fluids and temperatures, see the corresponding tables in the literature - experimentaly obtained.*

For a perfect (ideal) gass:

$$du = c_v dt \tag{1-8}$$

For $c_v = const$: $u_2 - u_1 = c_v (T_2 - T_1)$

SI units: J/kg

Specific entalpy, i:

Sum of the internal energy and energy due to the pressure change:

$$i = u + \frac{p}{\rho} \tag{1-9}$$

SI units: J/kg

For a perfect (ideal) gass:

$$d(u + \frac{p}{\rho}) = c_p dT$$

For $c_p = const$

$$(u + \frac{p}{\rho})_2 - (u + \frac{p}{\rho})_1 = c_p (T_2 - T_1)$$

∴ *For values for diferent fluids and temperatures, see the corresponding tables in the literature - experimentaly obtained.*

Compressibility and Bulk modulus of elasticity:

Compressibility β is a measure of the relative volume change of a fluid or solid as a response to a pressure (or mean stress) change:

$$\beta = -\frac{1}{V}\frac{dV}{dp} = +\frac{1}{\rho}\frac{d\rho}{dp} \tag{1-8}$$

The bulk modulus of elasticity is defined as reciprocal of compresibility:

$$E_V = \frac{1}{\beta} = -\frac{dp}{dV/V} = +\frac{dp}{d\rho/\rho} \tag{1-9}$$

The sign (-) shows that for $p \uparrow \;\Rightarrow\; V \downarrow$

SI units for E_V : N/m^2

∴ *Liquids (incompressible fluids) have large values of* E_V .

∴ *For values for diferent fluids see the corresponding tablesand diagrams in the literature*
For water $E_V = 2{,}06 \times 10^5$ N/cm^2

Velocity of sound, c:

Associated with each state of a substance, according Laplace formula:

$$c = \sqrt{dp/d\rho} = \sqrt{E_V/\rho} \qquad (1\text{-}10)$$

In case of *isentropic adiabatic process* of a gass: $\dfrac{p}{\rho^{\kappa}} = const$; $\kappa = \dfrac{c_p}{c_v}$

From (1-9) $\Rightarrow$ $E_v = \kappa p \quad \Rightarrow \quad c = \sqrt{\kappa p/\rho}$

For liquids c is determined from experimental values of E_V $\Rightarrow$ tables and dyagrams in the literature.

Vapor pressure, p_v - cavitation presuure, p_k:

Vapor pressure is the pressure of a vapor in equilibrium with its non-vapor phases.
All solids and liquids have a tendency to evaporate to a gaseous form, and all gases have a tendency to condense back.
At any given temperature, for a particular substance, there is a partial pressure at which the gas of that substance is in dynamic equilibrium with its liquid or solid forms. This is the vapor pressure of that substance at that temperature.

Cavitation (explained later on) = rapid (almost "explosive") change of of phase from liquid to vapor

∴ $p_k \approx p_v = f(liquidtype, T)$ $\Rightarrow$ *see the corresponding tables and dyagrams in the literature - experimentaly obtained.*

Surface energy and surface tension, σ:

At boundaries between gas and liquid phases or between different immiscible liquids, molecular attraction introduces forces which cause the interface to behave like a membrane under tension.

Surface tension is an effect within the surface layer of a liquid that causes that layer to behave as an elastic sheet.

This effect allows insects (such as the water strider) to walk on water. It allows small metal objects such as needles, razor blades, or foil fragments to float on the surface of water, and causes *capillary action*.

$$\sigma = \frac{force \times distance}{area} = \frac{work}{area} = \frac{force}{length}$$

Equations of state

An equation of state is a thermodynamic equation describing the state of matter under a given set of physical conditions = Dependance between the fluid properties.

Liquids

The equations of state for most physical substances are complex and are expressible in simple forms only for limited ranges of conditions.
True for liquids as well!
⇒ *use of tables and graphical curves obtained mostly experimentaly* ⇒ *empirical formula*
Important:
For wide range of pressures liquids are nearly incompressible.

Gases

For real gases and vapors ⇒ use of tables and graphical curves obtained mostly experimentaly ⇒ empirical formula.

For gases in a highly superheated condition a useful aproximation is *the theoretical equation of state for the perfect (ideal gas) - Clapeyron's equation:*

$$\frac{p}{\rho} = RT \tag{1-11}$$

An ideal gas or perfect gas is a hypothetical gas consisting of identical particles of zero volume, with no intermolecular forces.

T - absolute temperature in K (or ^{0}C); *p* - absolute pressure in N/m^2; ρ - *density* in kg/m^3;

R - gas constant in J/kgK (or J/kg^{0}C) - for dry air R = 287 J/kg^{0}C.

For ideal gas ⇒

$$c_p = c_v + R = \frac{\kappa}{\kappa - 1} R \, ; \qquad c_v = \frac{R}{\kappa - 1} ; \quad \kappa = \frac{c_p}{c_v} \tag{1-12}$$

c_p = specific heat at constant pressure;

c_v = specific heat at constant volume.

∴ *For values for diferent fluids see the corresponding tables in the literature*

Change of state proceses for gasses

Isothermal process:

$$\frac{p}{\rho} = RT = const \tag{1-13}$$

Constant pressure process:

$$p = \rho RT = const \tag{1-14}$$

Isentropic adiabatic process:

Zero heat transfer (adiabatic proces) and no friction (isentropic)

$$\frac{p}{\rho^{\kappa}} = const \tag{1-15}$$

$\kappa = \frac{c_p}{c_v}$ - adiabatic constant for the gas.

∴ *Additional equations can be obtained ⇒ see basic laws of Thermodynamics.*

1.4. Forces on a fluid element and pressure

Different kinds of forces acting on a fluid element with mass *m* in a volum *V*.
For example:

- *Mass (or volume) force - a force proportional to the mass of the fluid element ("body force"):*

  Gravity force (wight): $G = mg$

  Inertial force: $F_i = ma$

  Centrifugal force: $F_c = m\omega^2 r$ *et.c*

In general, a force per unit mass is defined: $\vec{R}$ in N/kg - *force per unit mass = "body force"*

$\therefore$ *The force on a mass dm is:*

$$d\vec{R} = dm\vec{R} = \rho dV\vec{R} \tag{1-16}$$

- *Force proportional to an area (area or surface force) - see Fig. 1.2:*

$$d\vec{S} = d\vec{P} + d\vec{T} = f(dA) \tag{1-17}$$

$d\vec{T}$ - *friction force*

$d\vec{T} = 0$ in case of *ideal fluid* and in case of *fluid at rest.* In that case $d\vec{S} = d\vec{P}$

$d\vec{P} = \vec{F}_p$ - *pressure force*

$\Rightarrow$ *pressure p:*

$$p = \frac{dP}{dA} = \frac{F_p}{A} \tag{1-18}$$

$\therefore$ *The pressure is a scalar property*

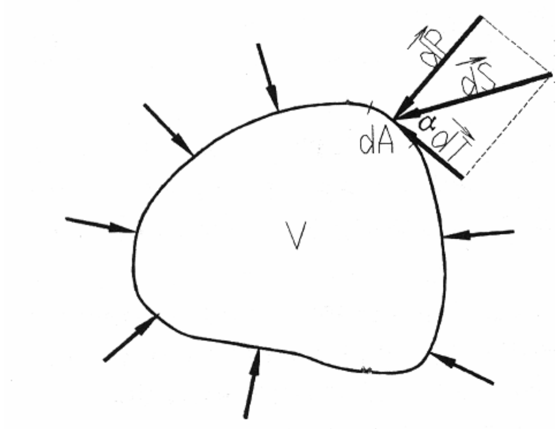


Fig. 1-2: Forces proportional to an area

- *Other forces acting in fluid flows: viscous forces, elastic forces, surface tension forces etc.*

2. Statics of fluids

2.1. Basic laws - hydrostatic pressure, Euler's equation of equilibrium

Hydrostatic pressure

The pressure exists in both cases of fluid at rest and flowing fluid ⇒ differences of the pressure characteristics.

In case of fluid at rest ⇒ hydrostatic pressure.

The same term (*hydro*) for compressible and incompressible fluid.

Two important characteristics of hydrostatic pressure:

- *it is always perpendicular to each surface in the fluid volume* - which is obvious following the pressure definition;
- *Its magnitude (value) doesn't change if that surface changes its position ⇒ the pressure value in one point is the same in all directions!*

⇒ *proof according Fig. 2.1:*

From the equilibrium of the surface forces on the supposed fluid tetrahedral element (with one corner at point *M*) in the directions *x,y,z* ⇒:

$$p_x dA_x - p dA \cos\alpha = 0$$

$$p_y dA_y - p dA \cos\beta = 0$$

$$p_z dA_z - p dA \cos\gamma = 0$$

Since from *Fig. 2.1*: $dA_x = dA\cos\alpha$; $dA_y = dA\cos\beta$; $dA_z = dA\cos\gamma$

⇒ $$p_x = p_y = p_z = p \qquad (2\text{-}1)$$

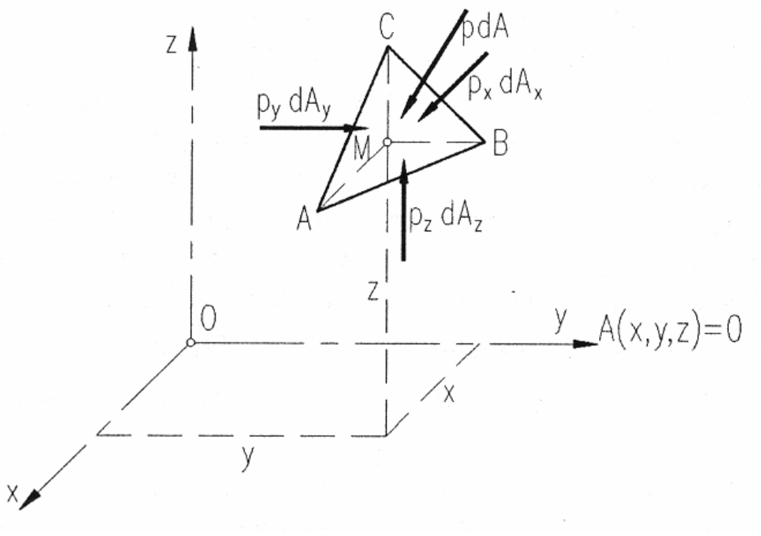


Fig. 2.1: Equilibrium of surface forces on a infinitesimal fluid element

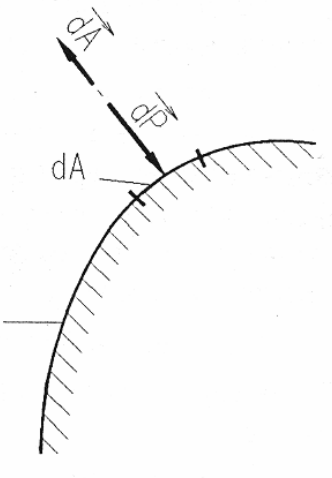


Fig. 2.2: Elementary pressure force

The elementary pressure force on an elementary surface $d\vec{A}$ (*Fig. 2.2*) as a vector is defined as:

$$d\vec{F}_p = d\vec{P} = -pd\vec{A} \tag{2-2}$$

⇒ *The resultant pressure force* over a certain surface A will be:

$$\vec{F}_p = -\int_A pd\vec{A} \tag{2-3}$$

Euler's equation of equilibrium

Elementary (infinitesimal) volume in the point M - $dV = dxdydz$ - *Fig. 2.3*.

Acting forces on the volume:

- *Pressure forces* $\vec{P}$
- *Elementary body force* $\vec{R}$ in N/kg

$$\vec{R} = X\vec{i} + Y\vec{j} + Z\vec{k} \tag{2-3}$$

X, Y *and* Z - components of $\vec{R}$ in x, y and z directions.

On the mass dm the total body force is:

$$\vec{R}dm = \vec{R}\rho dxdydz$$

in the "y" direction ⇒ $dmY = \rho Y dxdydz$

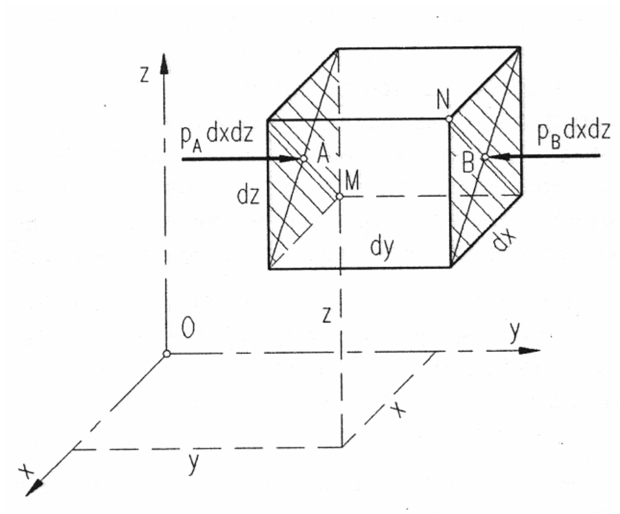


Fig. 2-3: Equilibrium of forces on an elementary volume

Equilibrium condition: $\vec{P} + \vec{R} = 0$ (2-4)

$\Rightarrow$ 3 scalar equations from the vector equation (2-4):

- in the "y" direction - see *Fig 2.3*:

$$p_A dxdz - p_B dxdz + \rho Y dxdydz = 0 \quad (2\text{-}5)$$

From *Fig. 2.3* $\Rightarrow$

In point M the pressure is: $p = p(x, y, z)$

in point A: $p_A = p + \frac{\partial p}{\partial x}\frac{dx}{2} + \frac{\partial p}{\partial z}\frac{dz}{2}$

in point B: $p_B = p + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial x}\frac{dx}{2} + \frac{\partial p}{\partial z}\frac{dz}{2}$

$\therefore$ The equation (2-5) is easily transformed into:

$$\rho Y dy = \frac{\partial p}{\partial y}dy \quad (2\text{-}5a)$$

- in the "x" direction: $\rho X dx = \frac{\partial p}{\partial x}dx$ (2-5b)

- in the "z" direction: $\rho Z dz = \frac{\partial p}{\partial z}dz$ (2-5c)

The equations (2-5a) to (2-5c) are known as *Euler's equation of equilibrium in scalar form*

The sum of the equations (2-5a) + (2-5b) + (2-5c) gives:

$$\rho(Xdx + Ydy + Zdz) = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz \quad (2\text{-}6)$$

where

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz \quad (2\text{-}7)$$

is *total pressure increase* from point *M(x,y,z)* to point *N(x+dx,y+dy,z+dz)*.

$\Rightarrow$

$$\rho(Xdx + Ydy + Zdz) = dp \quad (2\text{-}8)$$

(2-8) is *the fundamental equation of Static of fluids*

Force potential. Equipotential surfaces

Elementary body force $\vec{R} = \vec{R}(x, y, z) \Rightarrow$ $X = X(x, y, z)$; $Y = Y(x, y, z)$; $Z = Z(x, y, z)$

Barotropic fluid $\Rightarrow$ explicit function $\rho = \rho(p)$ $\Rightarrow$ (2-8) transforms into;

$$Xdx + Ydy + Zdz = \frac{dp}{\rho(p)} \quad (2\text{-}9)$$

$$dP = \frac{dp}{\rho(p)} \qquad (2\text{-}10)$$

$P = \int \frac{dp}{\rho(p)}$ - *generalized pressure.*

$\frac{\partial P}{\partial x} = \frac{1}{\rho}\frac{\partial p}{\partial x}; \quad \frac{\partial P}{\partial y} = \frac{1}{\rho}\frac{\partial p}{\partial y}; \quad \frac{\partial P}{\partial z} = \frac{1}{\rho}\frac{\partial p}{\partial z}$

∴ (2-9) is transformed into:

$$Xdx + Ydy + Zdz = dP \qquad (2\text{-}11)$$

(2-11) can be integrated only if the left side is also a total differential of certain scalar function:

$$Xdx + Ydy + Zdz = dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz \qquad (2\text{-}12)$$

$X = \frac{\partial U}{\partial x}; \quad Y = \frac{\partial U}{\partial y}; \quad Z = \frac{\partial U}{\partial z}$ - *components of the resultant body force* $\vec{R} = \vec{R}(x, y, z)$

$U = U(x, y, z)$ - *potential of the force, or potential function.*

⇒ (2-8) is transformed into:

$$\rho\left(\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz\right) = dp \qquad (2\text{-}13)$$

$$\rho dU = dp \qquad (2\text{-}13a)$$

Equipotential surface = surface on which $\rho dU = dp = 0 \quad \Rightarrow \quad dU = 0$

By integration it is obtained:

$U = U(x, y, z) = const; \quad p = p(x, y, z) = const$ *on a equipotential surface*

2.2. Equilibrium in gravity field

Fluid element at rest in a gravity field ⇒ only gravity force as G=mg

⇒ the only body force is in the "z" direction:

$$X = Y = 0; \quad Z = \frac{\partial U}{\partial z} = -g \qquad \text{N/kg}$$

∴ (2-13) is transformed into:

$$\rho(Xdx + Ydy + Zdz) = -\rho g dz = dp \qquad \text{i.e.}$$

$$dp = -\rho g dz = -\gamma dz \qquad (2\text{-}14)$$

The equation (2-14) = fundamental equation of equilibrium of fluid at rest in a gravity field.

Since $X = Y = 0$; and $Z = \frac{\partial U}{\partial z} = -g \Rightarrow \quad \frac{dU}{dz} = -g$

$$\Rightarrow \qquad U = -\int g dz + U_0 = -gz + U_0 \tag{2-15}$$

The equipotential surfaces (with $U = const$; $p = const$) in this case are:

$$U = -gz + U_0 = const$$

$$\Rightarrow \qquad z = const \tag{2-16}$$

∴ *The equipotential surfaces in this case are surfaces parallel to the horizon.*

The integration of the differential equation (2-14), also gives:

$$\int_{p_0}^{p} dp = p - p_0 = -\int_{z_0}^{z} \gamma dz = \int_{z}^{z_0} \gamma dz = g\int_{z}^{z_0} \rho dz \tag{2-17}$$

The pressure difference is equal to the weight of the fluid column between the surfaces "z_0" and "z".

Incompressible fluid in gravity field

$\rho = const$; $\gamma = \rho g = const$

⇒ (2-17) transforms into:

$$p - p_0 = -\gamma \int_{z_0}^{z} dz = \gamma(z_0 - z) = \rho g (z_0 - z) \tag{2-18}$$

∴ The pressure at the *"z"* level will be:

$$p = p_0 + \gamma h = p_0 + \rho g h \tag{2-19}$$

$h = z_0 - z$ = *height of the liquid column between points M_0 and M (see Fig. 2.6).*

If the level z_0 is the free surface to the atmosphere (see *Fig. 2.6*) ⇒

$p_0 = p_a$ = *atmospheric (barometric) pressure;*

∴ The pressure at the level "*z*" (point *M*) will be:

$$p = p_a + \gamma h = p_a + \rho g h \tag{2-20}$$

where:

p - absolute pressure

$$If\ p > p_a \qquad \Rightarrow \qquad p_m = p - p_a \tag{2-21}$$

p_m - over-pressure (gauge pressure)

$$If\ p < p_a \qquad \Rightarrow \qquad p_v = p_a - p \tag{2-22}$$

p_v - vacuum (sub-pressure or negative gauge pressure).

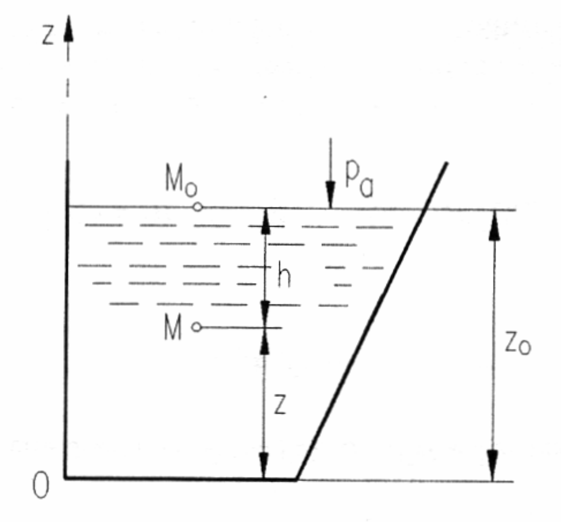


Fig. 2.6: Liquid with free surface in gravity field

Hydrostatic manometers

***Interconnected vessels** ⇒ see Fig. 2.7*

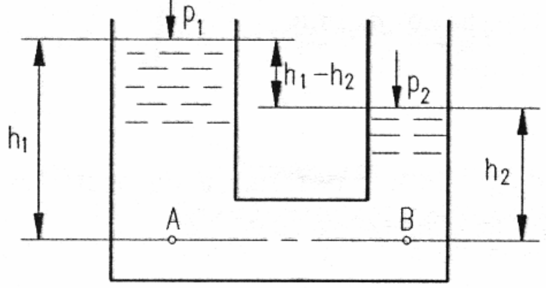


Fig. 2.7: Interconnected vessels

Every horizontal plane is a equipotential surface

The points A and B are laying on the same horizontal plane = *equipotential surface*

$\Rightarrow \quad p_A = p_B$

From the equation (2-19) ⇒

$$p_A = p_1 + \gamma h_1 \quad \text{and} \quad p_B = p_2 + \gamma h_2$$

Since $p_A = p_B \Rightarrow$

$$p_2 - p_1 = \gamma(h_1 - h_2) = \gamma h = \rho g h \tag{2-23}$$

Special case: $p_1 = p_2$ (for example $p_1 = p_2 = p_a$) $\Rightarrow \quad h = 0$

⇒ ∴ *In open interconnected vessels the free surfaces are laying in one horizontal plane*

Hydrostatic manometer, U-tube

Special case of interconnected vessels = an *instrument for measurement of gauge pressure* p_m

From *Fig. 2.8* ⇒ $p_2 = p$ *and* $p_1 = p_a$ ⇒

$$p_m = p - p_a = \gamma h = \rho g h \tag{2-24}$$

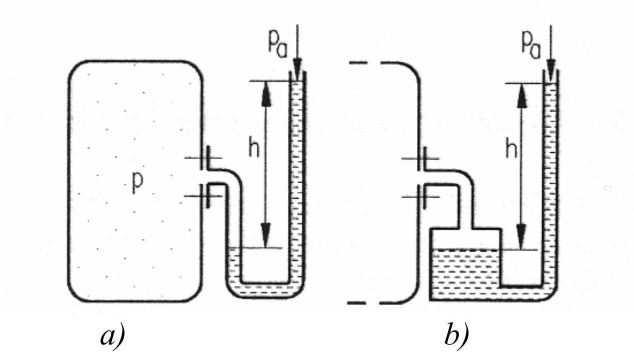


Fig. 2.8: Hydrostatic manometer; a) U-tube and b) Vessel manometer

The hydrostatic manometer can be used for *vacuum measurement* as well (*vacuum-meter*):

⇒ $p < p_a$ ⇒ $h < 0$ - the column level moves in the opposite direction (downwards).

$$p_v = p_a - p = |\rho g h|$$

A variety construction of U-tube is the *well-type (single-leg) manometer* (see *Fig. 2.8b)*).

Barometer

A variety of the single-leg manometer = instrument for measurement of *atmospheric (barometric) pressure* p_a, see *Fig. 2.9*:

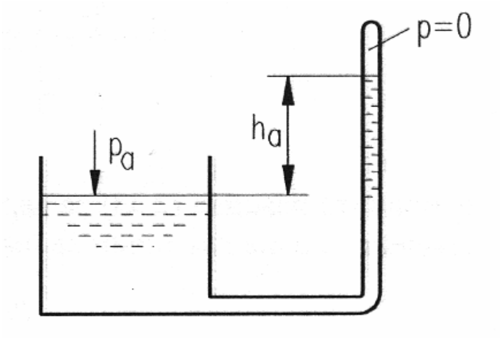


Fig. 2.9: Barometer Principle

In this case:

$p_1 = 0$ - the air is completely evacuated from the tube (leg).

$p_2 = p_a$ - the pressure in the well is *atmospheric (barometric) pressure*.

$$p_2 - p_1 = p_a = \gamma h_a = \gamma h = \rho g h_a \tag{2-25}$$

On the *sea level* at normal conditions and temperature $t = 15$ ^{0}C :

⇒ $p_a = 10{,}1325 \times 10^4$ N/m^2 = 1,01325 bar = 1 atm = *standard (physical) atmosphere.*

$\Rightarrow\ h_a = p_a / \gamma = 760$ mmHg - barometric liquid is mercury with $\gamma = \rho g = 13{,}3 \times 10^4$ N/m^3 ;
$\Rightarrow\ h_a = p_a / \gamma = 10{,}33$ mWC - barometric liquid is *water column* with $\gamma = \rho g = 9810$ N/m^3.

Pascal's Law

For equilibrium of liquid at rest in gravity field in two interconnected vessels (*Fig. 2.10*) $\Rightarrow$

$$p_B - p_A = \gamma(h_A - h_B) \tag{2-26}$$

If in point A, p_A is increased with value Δp_A (e.g. $\Delta p_A = \frac{4P_1}{\pi D^2}$, produced by the force on the piston, P_1) $\Rightarrow$ in the point B $\Delta p_B = ?$

For the fluid at rest in gravity field $\Rightarrow$

$$(p_B + \Delta p_B) - (p_A + \Delta p_A) = \gamma(h_A - h_B) \tag{2-27}$$

From the equations (2-26) and (2-27) $\Rightarrow$

$$\Delta p_A = \Delta p_B \tag{2-28}$$

$\therefore$ *the Law of Blaise Pascal:*

The pressure change in one point in liquid at rest in gravity field is transferred equally in all liquid points (on the container wals as wel).

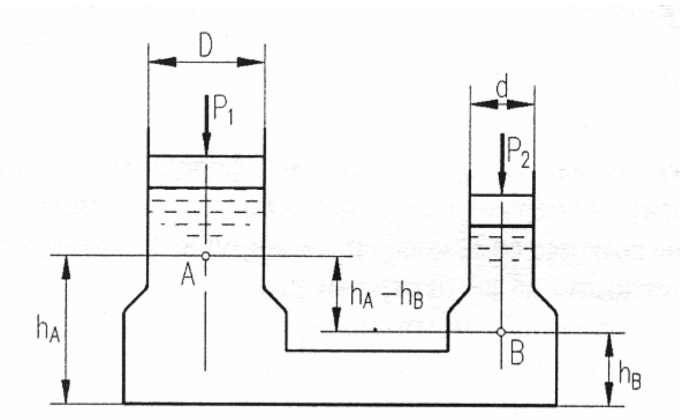


Fig. 2.10: Ilustration of the Pascal's Law

Example of aplication - **Hydraulic press** *(Fig. 2.11):*

If the acting force on the piston K_2 is $P_2 = \frac{a}{b} P$

$$\Rightarrow\ \Delta p_2 = \frac{4P_2}{\pi d^2}$$

According Pascal's Law $\Rightarrow$ the pressure on the piston K_1, $\Delta p_1 = \Delta p_2$

$$\Rightarrow\qquad \Delta p_1 = \frac{4P_1}{\pi D^2} = \frac{4P_2}{\pi d^2} = \frac{4}{\pi d^2}\frac{a}{b}P$$

$$\Rightarrow\qquad P_1 = \left(\frac{D}{d}\right)^2 \frac{a}{b} P$$

∴ *Depending the ratios D/d and a/b, higher force P_1 can be performed with smaller force P acting on the arm.*

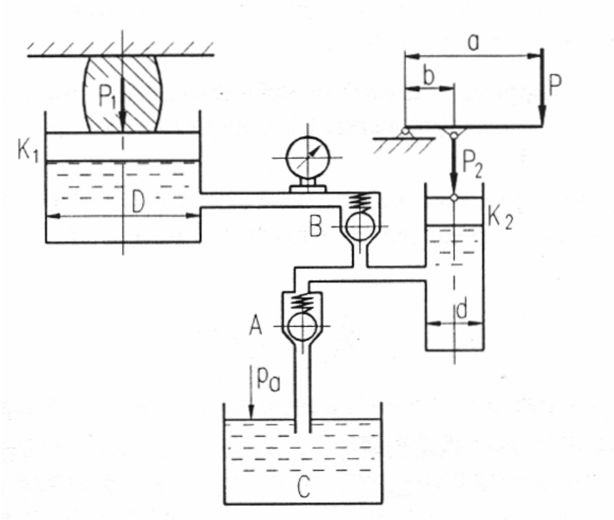


Fig. 2.11: Ilustration of a hydraulic press

Relative equilibrium of fluid

Acting forces on a fluid in rest: *Gravity forces* and other *Body forces*

In case of *gravity forces only* ⇒ the *free surface is horizontal* (normal to the gravity force).

In case of *relative equilibrium* (e.g. liquid at rest in moving container) ⇒ gravity forces, inertial forces, centrifugal forces etc).

Example: **Translation of a liquid container**

- *The container has a linear movement with constant velocity* ⇒ the free surface is also horizontal; since the gravity force is acting only.
- *The container has a linear accelerated movement with constant acceleration* $a = const$ (see *Fig. 2.12*):

two forces are acting: $G = mg$ *- gravity force; and* $F_i = -ma$ *- inertial force.*

⇒ *The free surface is normal to the resultant force* $F_R = ma'$ (see *Fig. 2.12*).

∴ *The equipotential surfaces as well as the free surface are normal to the resultant force F_R.*

If $a \neq const$ the liquid will oscillate in the container (the free surface will oscillate too).

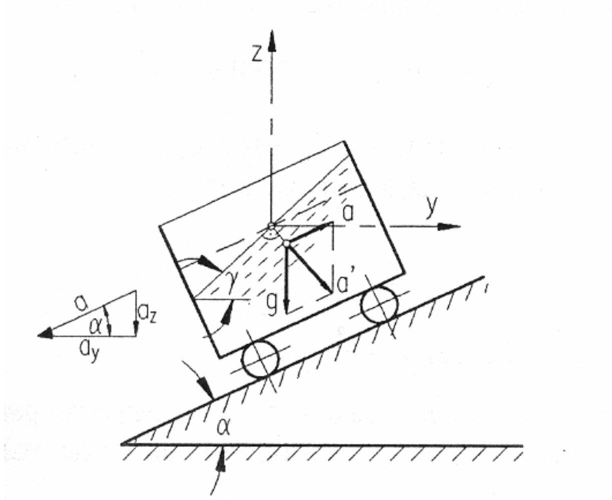


Fig. 2.12: Container with linear movement with constant acceleration

The equations of the equipotential surfaces (with *p = const*) can be obtained analytically from the fundamental equation of statics of fluids (2-8):

$$\rho(Xdx + Ydy + Zdz) = dp \tag{2-8}$$

From *Fig. 2.12* ⇒

$$a_x = 0; \qquad a_y = -a\cos\alpha; \qquad a_z = -a\sin\alpha$$

$$X = -a_x = 0; \quad Y = -a_y = a\cos\alpha; \quad Z = -a_z - g = a\sin\alpha - g$$

For *equipotential surfaces* $dp = 0$ in the equation (2-8) ⇒:

$$\Rightarrow \qquad a\cos\alpha dy + (a\sin\alpha - g)dz = 0 \tag{2-29}$$

The integrating of (2-29) gives:

$$ya\cos\alpha + (a\sin\alpha - g)z = C \tag{2-30}$$

At the free surface (*Fig. 2.12*), $y = z = 0 \Rightarrow C = 0$, from (2-30) ⇒

$$ya\cos\alpha + (a\sin\alpha - g)z = 0 \tag{2-31}$$

∴ *The free surface is a plane with an angle toward the horizontal obtained from:*

$$\tan\gamma = \frac{a\cos\alpha}{g - \sin\alpha} \tag{2-32}$$

The pressure change is obtained from (2-8):

$$\Rightarrow \qquad \rho\left[a\cos\alpha dy + (a\sin\alpha - g)dz\right] = dp \tag{2-33}$$

The integrating gives:

$$\rho\left[ya\cos\alpha + (a\sin\alpha - g)z\right] = p + C_1$$

At the coordinates beginning point at the free surface: $y = z = 0$, and $p = p_a \quad \Rightarrow \quad C_1 = -p_a$

$$\Rightarrow \qquad p = \rho\left[ya\cos\alpha + (a\sin\alpha - g)z\right] + p_a \tag{2-34}$$

Example: ***Rotation of a liquid container around vertical axis***

Rotation of a container with $\omega = const \Rightarrow$ rotation of the liquid together with the container as a whole, as on the *Fig 2.13*

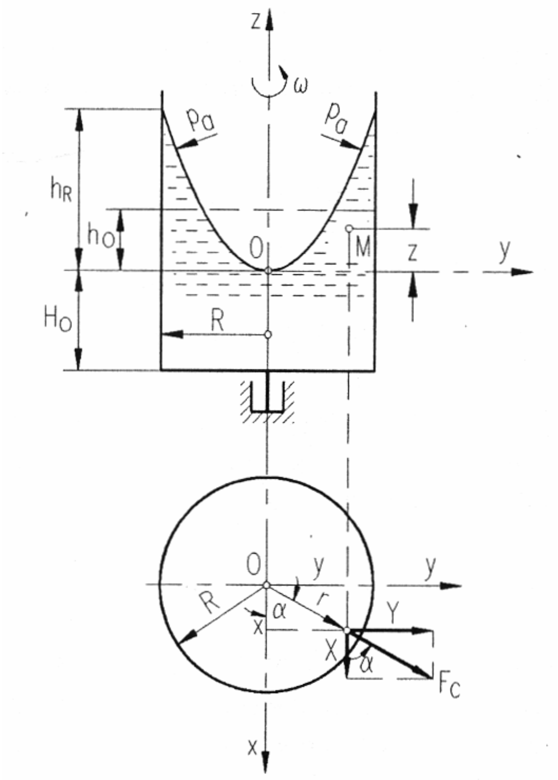


Fig. 2.13: Rotation of a liquid container around vertical axis

$\Rightarrow$ *The liquid is in relative rest (equilibrium)*

$\Rightarrow$ Acting forces per unit mass: $Z = -g$, $F_c = r\omega^2$

$\Rightarrow$ Forces per unit mass in *x* and *y* direction:

$$X = F_c \cos\alpha = \omega^2 r \cos\alpha = \omega^2 x \,; \qquad Y = F_c \sin\alpha = \omega^2 r si\alpha = \omega^2 y$$

For *equipotential surfaces* $dp = 0$ in the equation (2-8) $\Rightarrow$:

$$\omega^2 xdx + \omega^2 ydy - gdz = 0$$

The integrating gives:

$$\frac{1}{2}\omega^2\left(x^2 + y^2\right) - gz = C$$

At the coordinates beginning point at the free surface: $x = y = z = 0$, $\Rightarrow C = 0$

$$\Rightarrow \qquad \frac{1}{2}\omega^2\left(x^2 + y^2\right) - gz = 0 \qquad \Rightarrow \qquad z = \frac{\omega^2}{2g} r^2 \tag{2-35}$$

which is a rotating paraboloid.

The pressure change can be also obtained from the equation (2-8) $\Rightarrow$

$$p = p_a + \left[\frac{1}{2}\omega^2\left(x^2 + y^2\right) - gz\right] = p_a + \rho\left(\frac{1}{2}\omega^2 r^2 - gz\right) \tag{2-36}$$

Pressure force on a flat and curved surface

Pressure force on a flat surface

In general the pressure force:

$$\vec{P} = -\int_A p d\vec{A} \tag{2-37}$$

On a flat surface *Fig. 2.14*:

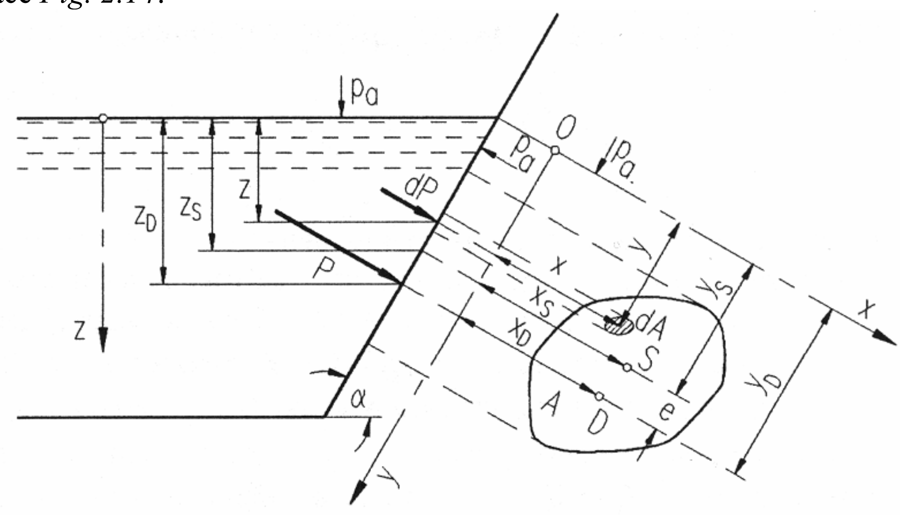


Fig. 2.14: Pressure force on a flat surface

Elementary force *dP*:

$$dP = (p - p_a)dA = \rho gz dA \tag{2-38}$$

$$\Rightarrow \qquad P = \rho g \int_A z dA \tag{2-39}$$

The integral is the *static moment* of the surface *A* related to the free surface.

S = center of mass (gravity)

D = acting point of the pressure force; $D \neq S$

$$\Rightarrow \qquad A z_S = \int_A z dA$$

$$\Rightarrow \qquad P = \rho g A z_S = \gamma A z_S \tag{2-40}$$

Coordinates of D:

From the equations: $P x_D = \int_A x dP$ and $P y_D = \int_A y dP$ (from the *theoreme of Varignon*),

$$\Rightarrow \qquad A y_S x_D = \int_A xy dA \qquad A y_S y_D = \int_A y^2 dA$$

$$\Rightarrow \qquad x_D = \frac{J_{xy}}{A y_S} \qquad y_D = \frac{J_x}{A y_S} \tag{2-41}$$

J_{xy} - centrifugal moment of inertia related to x and y axis;

J_x - centrifugal moment of inertia related to x axis.

$$\Rightarrow \qquad e = y_D - y_S = \frac{J_{xo}}{A y_S} \tag{2-42}$$

$J_x = J_{xo} + A y_S^2$ - from *Steiner's theorem*

J_{xo} - *proper moment of inertia* (moment of inertia about the centre of gravity *S*),

Example: Fig. 2.15 ($y_s = z_z$, concerning eq. (2-40))

- *for rectangular cover:*

$$y_S = y_O + \frac{a}{2}\ ; \qquad P = \rho g a b\left(\frac{a}{2} + y_O\right); \qquad e = \frac{a^2}{6(a + 2y_O)}$$

- *for circular cover:*

$$y_S = y_O + \frac{d}{2}; \qquad P = \frac{1}{4}\rho g d^2 \pi\left(\frac{d}{2} + y_O\right); \qquad e = \frac{d^2}{16(\frac{d}{2} + y_O)}$$

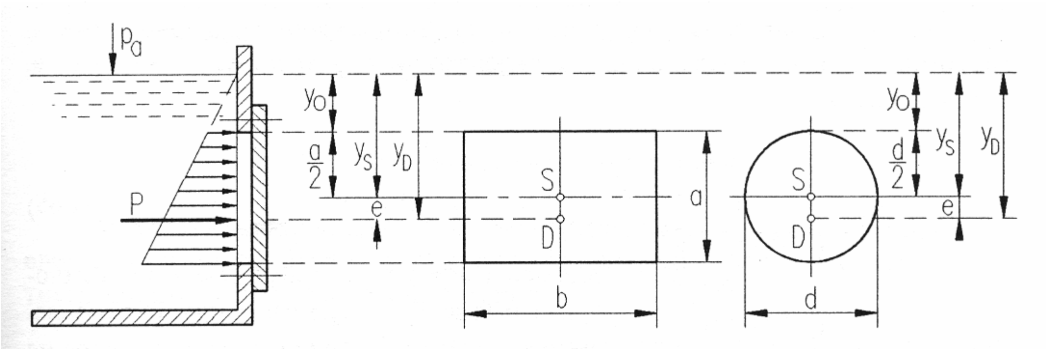


Fig. 2.15: Example - pressure force on a flat surface

Horizontal and vertical components of a pressure force - Fig. 2.16:

$$P_H = P\sin\alpha\ ; \qquad P_V = P\cos\alpha = \rho g V \tag{2-43}$$

$P = \gamma z_S A$; $\quad P_V = \gamma z_S A\cos\alpha$; $\quad V = z_S A\cos\alpha$; $\quad \gamma = \rho g$

V = volume of the liquid column acting on the surface *A* (see *Fig. 2-16*).

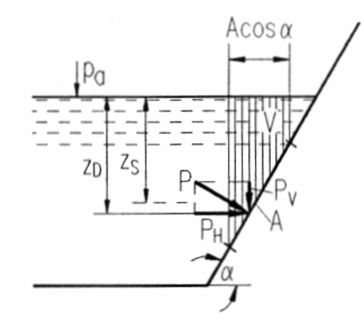


Fig. 2.16: Horizontal and vertical components of a pressure force

Pressure force on horizontal bottom - Fig. 2.17:

$p - p_a = \gamma h$ - over-pressure on any point of the bottom.

$$\Rightarrow \qquad P_V = \int_A p dA = \gamma h \int_A dA = \gamma h A = \gamma V$$

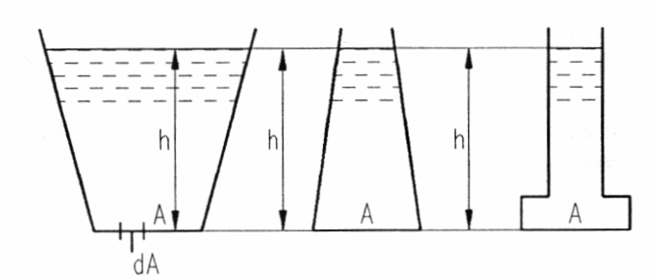


Fig. 2.17: Pressure force on horizontal bottom

Pressure force on a curved surface - *Fig. 2.18:, Fig. 2.19:, Fig. 2.21:*

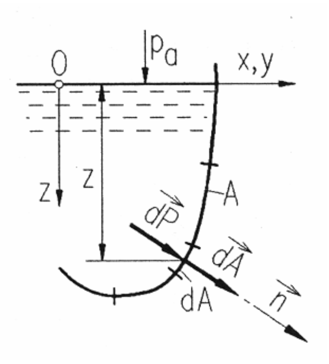
$$dP_x = dP\cos\alpha = \rho gzdA\cos\alpha = \rho gzdA_x$$
$$dP_y = dP\cos\beta = \rho gzdA\cos\beta = \rho gzdA_y \qquad (2\text{-}44)$$
$$dP_z = dP\cos\gamma = \rho gzdA\cos\gamma = \rho gzdA_z$$

$\Rightarrow$

$$P_x = \rho g \int_{Ax} zdA_x = \rho g z_{Sx} A_z$$
$$P_y = \rho g \int_{Ay} zdA_x = \rho g z_{Sy} A_y \qquad (2\text{-}45)$$
$$P_z = \rho g \int_{Az} zdA_z = \rho g \int_V dV = \rho g V$$

V = volume of the liquid column acting on the curved surface A (see *Fig. 2-21*).

P_z - weight of the liquid column acting on the curved surface.



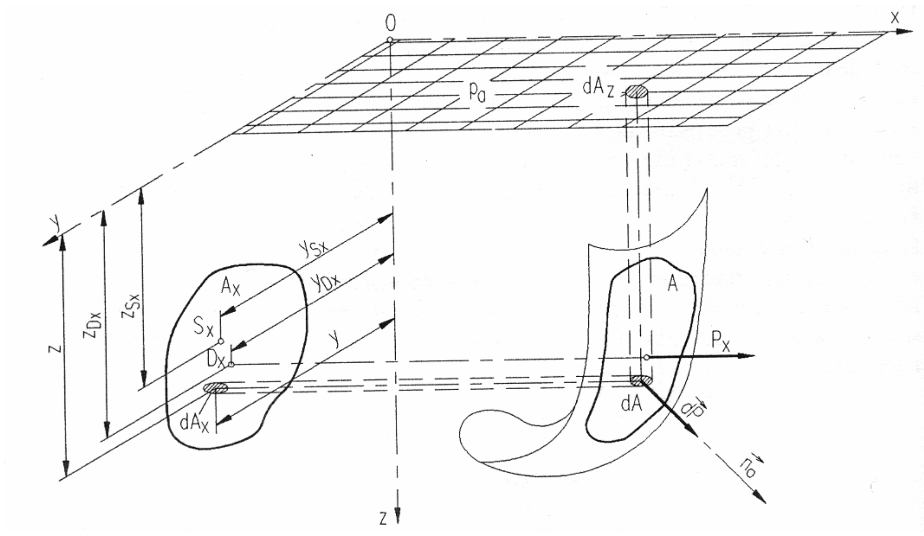


Fig. 2.18: *Fig. 2.19:*

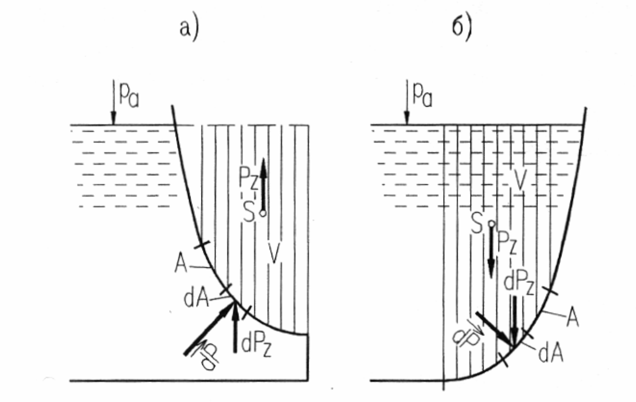


Fig. 2.21:

Buoyant forces

Buoyancy is the upward force on an object produced by the surrounding fluid (i.e., a liquid or a gas) in which it is fully or partially immersed, due to the pressure difference of the fluid between the top and bottom of the object.

The net upward buoyancy force is equal to the magnitude of the weight of fluid displaced by the body. This force enables the object to float or at least to seem lighter. Buoyancy is important for many vehicles such as boats, ships, balloons, and airships.

Acting forces in *x dirrection* (*fig. 2.23*):

$$dP_{x1} = dP_{x2} = \rho gzdA_x$$

Acting forces in vertical *z dirrection* (*fig. 2.23*):

$$dP_z = dP_{z2} - dP_{z1} = \rho g(z_2 - z_1)dA = \rho gdV$$

dV - volume of the elementary vertical cylinder.

⇒ *the buoyant force* or *Archimed's force P_z*:

$$P_z = \gamma \int_V dV = \gamma V = \rho gV \qquad (2\text{-}46)$$

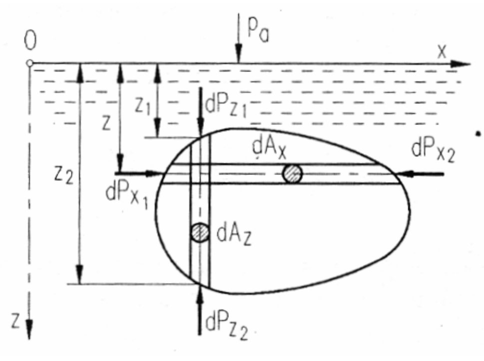


Fig. 2.23: Acting forces on an immersed body

Equation of floating - *see Fig. 2.24 and Fig. 2.25*

G - proper weight (gravitational force) of the body.

S - acting point of G (center of gravity).

D - acting point of P_z.

$\therefore$ *The body is floating (Fig. 2.24b) if:*

- $P_z = G$
- *the points S and D are on a same vertical line.*

$\therefore$ *If S and D are not on a same vertical line, the body rotates until S and D reach the same line. (see Fig. 2.25).*

$\therefore$ *If $G > P_z$, the body is sinking downwards.*

$\therefore$ *If $G < P_z$, the body is moving upwards until reaches the free surface (floating condition).*

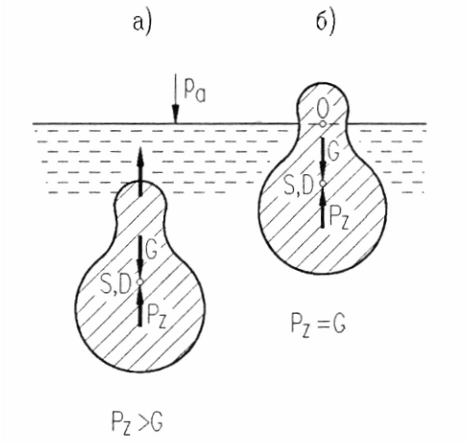


Fig. 2.24:

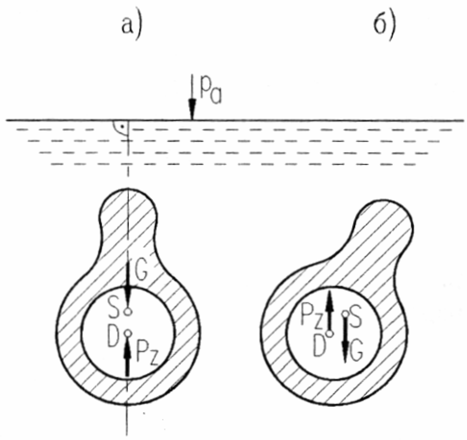
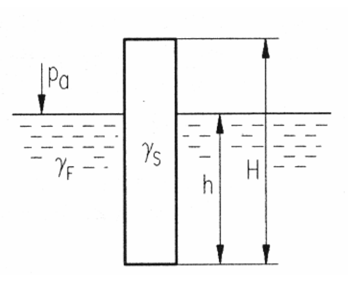


Fig. 2.25

Example - prismatic body (Fig. 2.26)

 - *Principle of areometer (hydrometer)* - instrument for measuring density/specific weight of a fluid.



$$\gamma_S AH = \gamma_F Ah \;\Rightarrow\; h = \frac{\gamma_S}{\gamma_F} H = \frac{\rho_S}{\rho_F} H \;\Rightarrow\; \rho_F = \frac{\rho_S H}{h}$$

ρ_F *- density of the fluid;* ρ_S *- density of the body*

Fig. 2.26

3. Kinematics of flow

3.1. Flow field - velocity field

Fluid flow - movement of the fluid particles.
Every particle in the fluid has different properties - difference with solid body movement.

Flow field - change of fluid flow properties (properties in every point) in space and time.

Velocity and acceleration of a fluid particle are vectors (see Fig.3.1).

∴ ***Velocity field*** *- a vector field, which is used to mathematically describe the motion of the fluid.*

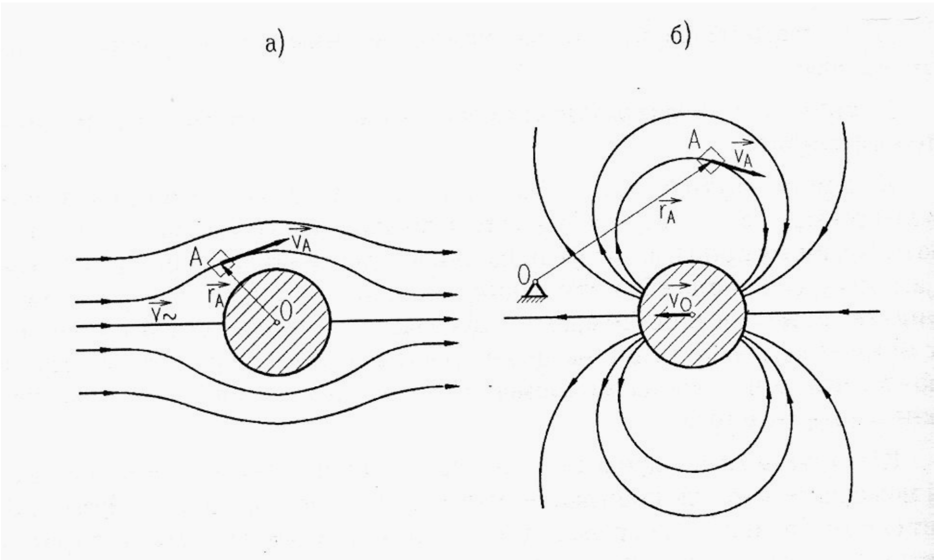


Fig.3.1: Examples of flow field

⇒ *two approaches for flow field defining - Lagrangian and Eulerian approach*:

Lagrangian approach:

A point in the space determined with a position vector $\vec{r}_L$ corresponds to every fluid particle with mass *dm* at certain time $t = 0$.

∴ *The position of the fluid particle* $\vec{r}$ *is defined as:*

$$\vec{r} = \vec{r}(\vec{r}_L, t) \qquad (3\text{-}1)$$

$\vec{r}_L$ *- position vector, Fig. 3.1*

∴ *The coordinates of particles are represented as functions of time!*

With application of the *Newton's equation* to the fluid particle ⇒ *Lagrangian differential equations*.
⇒ very difficult solving ⇒ *the application is not practical*.
The description of the entire flow field is essentially an instantaneous picture of the velocity and acceleration of every particle.

Eulerian approach:

Approach with significant advantage.

∴ *The particle velocities at various points are given as functions of time:*

$$\vec{v} = \vec{v}(\vec{r}, t) \tag{3-2}$$

Similar expressions for any fluid flow property can be defined; e.g.:

$$\vec{f} = \vec{f}(\vec{r}, t) \tag{3-3}$$

∴ *If the functions as (3-3) are defined for all fluid flow properties ⇒ the fluid flow field is completely solved.*

Steady (stationary) flow: $\frac{\partial \vec{v}}{\partial t} = 0 \Rightarrow \vec{v} = \vec{v}(\vec{r})$

Unsteady flow: $\frac{\partial \vec{v}}{\partial t} \neq 0$

3.2. Velocity, streamlines and path lines, stream function, stream tube, velocity gradient and shear

Velocity $\vec{v}$ *is defined as a vector dependent of the position vector* $\vec{r}$ *of a point (particle) in the flow space and time* - see *Fig. 3.1* and *Fig. 3.2*:

$$\vec{v} = \vec{v}(\vec{r}, t) \tag{3-4}$$

In 3-D *Cartesian* (*Descartes*) coordinate system - see *Fig. 3.2* and *Fig. 3.3a*:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{s}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k} \tag{3-5}$$

$$d\vec{s} = d\vec{r} = dx\ \vec{i} + dy\ \vec{j} + dz\ \vec{k}$$

s - length along stream line *(path)*; $d\vec{s}$ - *elementary path*

velocity components: $v_x = \frac{dx}{dt}$; $v_y = \frac{dy}{dt}$; $v_z = \frac{dz}{dt}$

velocity intensity:

$$\left|\vec{v}\right| = \sqrt{v_x^2 + v_y^2 + v_z^2} \tag{3-6}$$

In *polar* coordinate system - see *Fig. 3.3a)*:

$$x = r\cos\theta\,; \qquad y = r\sin\theta \tag{3-7}$$

$$v_r = v_x\cos\theta + v_y\sin\theta\,; \quad v_\theta = v_y\cos\theta - v_x\sin\theta \tag{3-8a}$$

$$v_x = v_r\cos\theta - v_\theta\sin\theta\,; \quad v_y = v_\theta\cos\theta + v_r\sin\theta \tag{3-8b}$$

The axisymmetric flow (*Fig. 3.3b*) is defined only with the components the components v_r and v_z:

$$\Rightarrow \qquad v_x = v_r \cos\theta \; ; \;\; v_y = v_r \sin\theta \;\; ; \;\; v_z = v_z \tag{3-9}$$

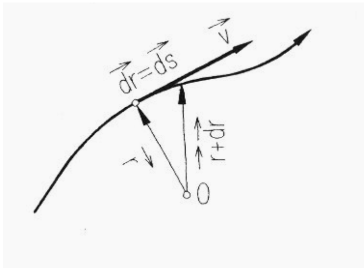


Fig. 3.2: Velocity vector

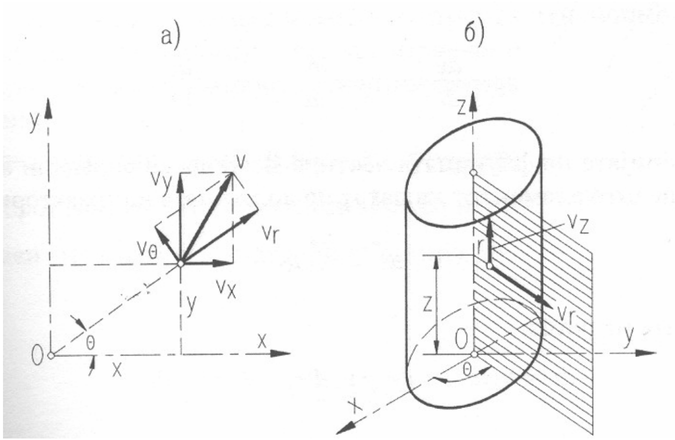
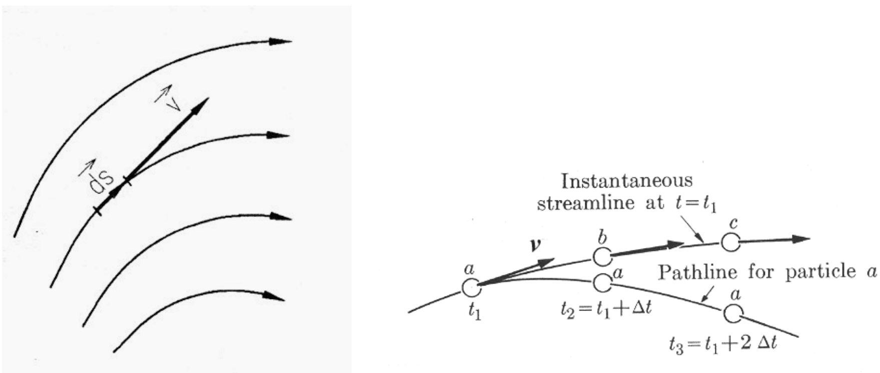


Fig. 3.3: Cartesian versus polar (cylindrical) system and axisymmetric flow

Streamlines versus pathlines*:*

Streamlines are a family of curves that are instantaneously tangent to the velocity vector of the flow - see *Fig. 3.4.*
Streamline - in every point the dirrection of the velocity is identical with the tangent line in that point:

$$\Rightarrow \qquad \vec{v} = \lambda \vec{ds} \tag{3-10}$$



a) *b)*

Fig. 3.4: Streamlines and path lines

since: $\vec{v} = v_x \vec{i} + v_y \vec{j} + v_z \vec{k} \Rightarrow$

$$\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z} = \frac{1}{\lambda} \tag{3-11}$$

For 2-D flow (*Fig. 3.5*) ⇒

$$\frac{dy}{dx} = \frac{v_y}{v_x} = \tan\alpha \qquad (3\text{-}12)$$

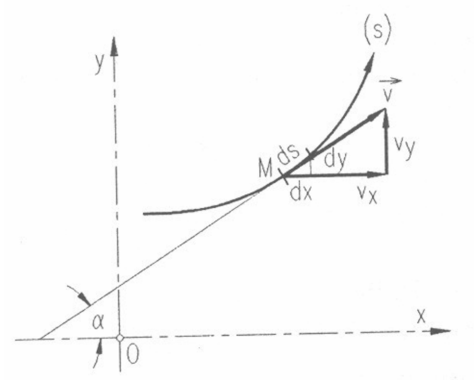


Fig. 3.5: Slope of a streamline

Pathlines are the trajectory that a fluid particle would make as it moves around with the flow. In unsteady flow, the fluid particle will not, in general, remain on the same stream line (see *Fig. 3.4b*).

∴ *In steady motion streamlines are the same as pathlines.*

Stream function

For 2-D flows streamlines definition *a stream function* $\psi(x, y)$ is defined!
The velocity components are defined with this function as:

$$v_x = \frac{\partial \psi}{\partial y}; \qquad v_y = -\frac{\partial \psi}{\partial x} \qquad (3\text{-}13)$$

⇒ The differential equation for the stream line (3-12) becomes:

$$v_x dy - v_y dx = 0 \qquad (3\text{-}14a)$$

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = d\psi = 0 \qquad (3\text{-}14b)$$

⇒ along a stream line $\psi(x, y) = const$ (3-15)

Stream tube

A *stream tube* or *stream filament* is a small imaginary tube or "conduit" bounded by streamlines. Because the streamlines are tangent to the flow velocity, fluid that is inside a stream tube must remain forever within that same stream tube (see *Fig. 3.6*).

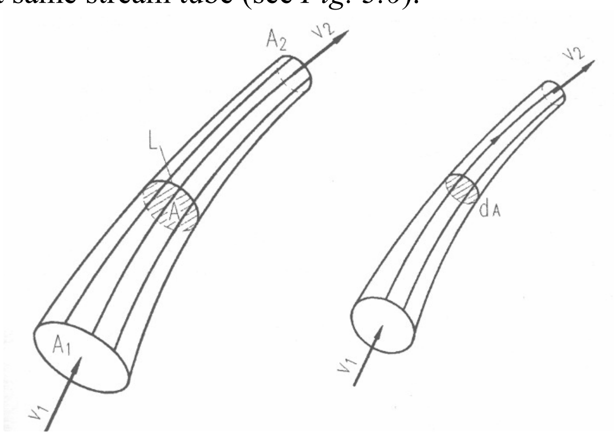


Fig. 3.6: Stream tube

Velocity gradients and shear

The *velocity change* in the vicinity of a point can be expressed in terms of the *partial derivatives* of the four independent variables (*x,y,z,t*).

⇒ *velocity change in the x-direction*:

$$dv_x = \frac{\partial v_x}{\partial t}dt + \frac{\partial v_x}{\partial x}dx + \frac{\partial v_x}{\partial y}dy + \frac{\partial v_x}{\partial z}dz \qquad (3\text{-}16)$$

Velocity changes in *y* and *z* directions can be expressed with similar expressions to (3-16).

The *rate of change* of the velocity *in the x-direction* (*total derivative*) is:

$$\frac{dv_x}{dt} = \frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z} \qquad (3\text{-}17)$$

$\frac{\partial v_x}{\partial t}, \frac{\partial v_x}{\partial x}, \frac{\partial v_x}{\partial y}, \frac{\partial v_x}{\partial z}$ …… - *velocity gradients*

$\frac{\partial v_x}{\partial t}$ = *"local" change;*

$v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}$ = *"convective" change.*

Any other property of the fluid or its motion can be treated in this way. For example, *the total rate of density change* for for compressible fluid:

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + v_x\frac{\partial \rho}{\partial x} + v_y\frac{\partial \rho}{\partial y} + v_z\frac{\partial \rho}{\partial z}$$

⇒ *acceleration components*:

$$a_x = \frac{dv_x}{dt} = \frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}$$
$$a_y = \frac{dv_y}{dt} = \frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z} \qquad (3\text{-}18)$$
$$a_z = \frac{dv_z}{dt} = \frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}$$

⇒ *steady flow* - if all *local accelerations* are zero.
⇒ *uniform flow* - if all *convective accelerations* are zero.

Velocity gradients are also measures for rate of deformation!
For example ***the shear stress*** (equation (1-6) in Chapter 1.3):

$$\tau = \mu\frac{dv}{dn}$$

3.3. Volume flow, flux and circulation

The ***volumetric flow rate***, or ***volume flow rate***, is the volume of fluid which passes through a given surface per unit time (for example [m^3/s] in SI units) - see *Fig. 3.7*.

For steady flow, from *Fig. 3.7* ⇒ $dV = dAdh = dAds\cos\alpha$; since $ds = vdt$ ⇒

⇒ *elementary volume flow rate:* $dQ = \frac{dV}{dt} = vdA\cos\alpha = v_n dA = \left(\vec{v}, d\vec{A}\right)$ (3-19)

$\left(\vec{v}, d\vec{A}\right)$ = scalar product of $\vec{v}$ and $d\vec{A}$.

The entire *volume flow rate through the given area A* see *Fig. 3.7* will be:

$$Q = \int_A dQ = \int_A \left(\vec{v}, d\vec{A}\right) = Av\cos\alpha \qquad (3\text{-}20)$$

If the flow is uniform and perpendicular to the area A ($\alpha = 90^0$) - i.e. $v \perp A$, and $v = const \Rightarrow$

$$Q = v\int_A dA = vA \qquad (3\text{-}20a)$$

Mass flow rate is the movement of *mass per time*. Its unit are [kg/s] in SI units:

$$\dot{m} = Q_m = \int_A \rho\left(\vec{v}, d\vec{A}\right) = \rho Q \qquad (3\text{-}21)$$

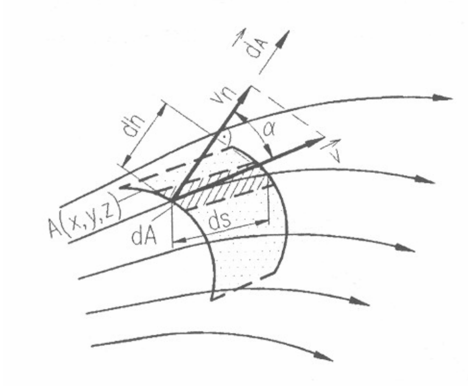


Fig. 3.7: Flow rate

Concerning *Fig. 3.7:*
$d\vec{A}$ - elementary area which moves with a velocity $\vec{v} \Rightarrow$ after time dt, the fluid particles on dA will make a path of $ds \Rightarrow dV = dAdh = dAds\cos\alpha$

Other properties connected to the velocity change can be defined:

Velocity flux (flow through the curve L) - see *Fig. 3.8*:

$$\Phi = \int_A^B \left(\vec{v}, \vec{dl}\right) \qquad (3\text{-}22)$$

Velocity circulation (along the closed curve L) - see *Fig. 3.8*:

$$\Gamma = \oint_L \left(\vec{v}, \vec{dl}\right) \qquad (3\text{-}23)$$

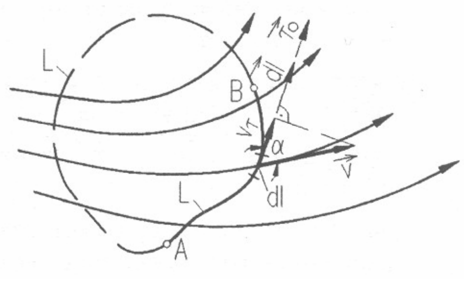


Fig. 3.8: Velocity flux and circulation

3.4. Continuity equations

Flow through a prismatic flow element - *Fig. 3.16* ⇒

- *velocity in the point M*: $\vec{v} = v_x \vec{i} + v_y \vec{j} + v_z \vec{k}$
- *velocity in "y" direction in the point A*: $v_A = v_y + \frac{\partial v_y}{\partial x}\frac{dx}{2} + \frac{\partial v_y}{\partial z}\frac{dz}{2}$
- *velocity in "y" direction in the point B*: $v_B = v_y + \frac{\partial v_y}{\partial y}dy + \frac{\partial v_y}{\partial x}\frac{dx}{2} + \frac{\partial v_y}{\partial z}\frac{dz}{2}$

The rate of volume flow change in *"y"* direction, for *incompressible fluid flow* $\rho = const$, is:

$$\delta Q_y = (v_A - v_B)dxdz = -\frac{\partial v_y}{\partial y}dxdydz = -\frac{\partial v_y}{\partial y}dV \qquad \text{(3-24a)}$$

On the same manner in the "*x*" and "*z*" directions ⇒

$$\delta Q_x = -\frac{\partial v_x}{\partial x}dV \;; \qquad \delta Q_z = -\frac{\partial v_z}{\partial z}dV \qquad \text{(3-24b)}$$

For *compressible fluid flow*, $\rho \neq const$, the rate of mass flow changes Q_{mx}, Q_{my}, Q_{mz} have to be treated ⇒:

$$\delta Q_{mx} = -\frac{\partial(\rho v_x)}{\partial x}dV \;; \; \delta Q_{my} = -\frac{\partial(\rho v_y)}{\partial y}dV \;; \; \delta Q_{mz} = -\frac{\partial(\rho v_z)}{\partial z}dV \qquad \text{(3-25)}$$

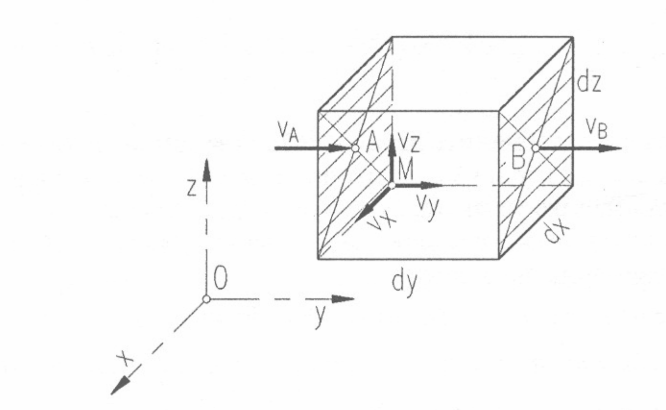


Fig. 3.16: Flow through a prismatic element

The total excess of the volume flow rate δQ will be:

$$\delta Q = \delta Q_x + \delta Q_y + \delta Q_z = -\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)dV = -dV\,\mathrm{div}\,\vec{v} \qquad \text{(3-26)}$$

$\mathrm{div}\,\vec{v} = \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)$ - *divergence of* $\vec{v}$.

∴ *The total excess of the mass flow rate* δQ_m (for compressible fluid) - excess of mass passing into the element per unit time:

$$\delta Q_m = -\left(\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}\right)dV = -dV\,\mathrm{div}(\rho\vec{v}) \qquad \text{(3-27)}$$

The principle of conservation of matter $\Rightarrow \quad \delta Q_m = \frac{\partial}{\partial t}(dm) = \frac{\partial}{\partial t}(\rho dV) = dV \frac{\partial \rho}{\partial t}$

Since dV is independent of time (the control volum dV is fixed) $\Rightarrow$ *The general continuity equation for unsteady flow of compressible fluid* :

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = 0 \qquad (3\text{-}28)$$

For *steady compressible fluid flow*, $\frac{\partial \rho}{\partial t} = 0 \Rightarrow$`

$$\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = 0 \qquad (3\text{-}29)$$

$\therefore$ *The continuity equation for incompressible fluid flow,* $\rho = const$:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \qquad (3\text{-}30)$$

For 2-D flow $\Rightarrow \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$

3.5. Acceleration

For 3-D flow:

$$\vec{a} = \frac{d\vec{v}}{dt} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k} \qquad (3\text{-}31)$$

Acceleration components - see (3-18):

$$\begin{aligned} a_x &= \frac{dv_x}{dt} = \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \\ a_y &= \frac{dv_y}{dt} = \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \\ a_z &= \frac{dv_z}{dt} = \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \end{aligned} \qquad (3\text{-}18)$$

For one-dimensional flow:

One dimensional gravity flow along a stream line "*s*" - see *Fig. 3.17* $\Rightarrow$

$v = v(s,t)$

$$dv = \frac{\partial v}{\partial t} dt + \frac{\partial v}{\partial s} ds \qquad (3\text{-}32)$$

$$a = \frac{dv}{dt} = \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial s} \qquad (3\text{-}33)$$

where: $ds = vdt$ is the *path of the fluid particle* along the streamline (see *Fig. 3.17*).

For *steady flow*, $\frac{\partial v}{\partial t} = 0 \quad \Rightarrow$

$$a = v \frac{\partial v}{\partial s} \qquad (3\text{-}34)$$

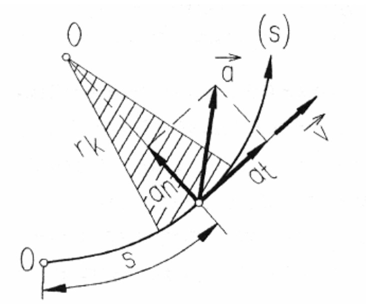


Fig. 3.17:One dimensional flow along a streamline

For 2-D flow:

$v_z = 0$ and $\frac{\partial}{\partial z} = 0$, from (3-31) and (3-18) $\Rightarrow$

$$\vec{a} = \frac{d\vec{v}}{dt} = a_x\vec{i} + a_y\vec{j} \tag{3-35}$$

$$a_x = \frac{dv_x}{dt} = \frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} \tag{3-36a}$$

$$a_y = \frac{dv_y}{dt} = \frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} \tag{3-36b}$$

For 2-D steady flow, $\frac{\partial v}{\partial t} = \frac{\partial v_x}{\partial t} = \frac{\partial v_y}{\partial t} = 0 \quad \Rightarrow$

$$a_x = v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} \tag{3-37a}$$

$$a_y = v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} \tag{3-37b}$$

In *polar* coordinate system - see *Fig. 3.3* and equations (3-8) and (3-36):

$$a_r = \frac{dv_r}{dt} = \frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} \tag{3-38a}$$

$$a_\theta = \frac{dv_\theta}{dt} = \frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} \tag{3-38b}$$

For coordinates starting point *O* at the curvature center of the streamline (*Fig. 3.18*) $\Rightarrow$

$r = r_k$; $v_r = 0$; $\Rightarrow$ $ds = r_k d\theta$ $v_\theta = v$;

$a_r = a_n$ (normal acceleration); $a_\theta = a_t$ (tangential acceleration):

$$a_n = -\frac{v^2}{r_k} \tag{3-39a}$$

$$a_t = \frac{\partial v}{\partial t} + v\frac{\partial v_\theta}{\partial s} \tag{3-39b}$$

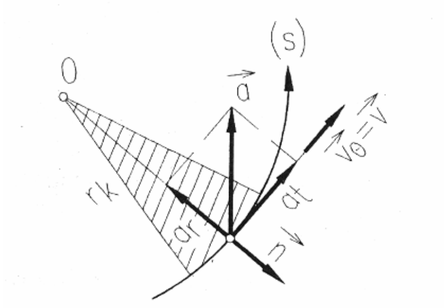


Fig. 3.18: Acceleration components in polar coordinates

For *steady axisymetric 2-D flow* (see *Fig. 3.3b* and equations (3-9)):

$\frac{\partial v}{\partial t} = 0$; $v_\theta = 0$ and $\frac{\partial}{\partial \theta} = 0 \Rightarrow$

$$a_r = v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} = \frac{dv_r}{dt} \quad (3\text{-}40a)$$

$$a_z = v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = \frac{dv_z}{dt} \quad (3\text{-}40b)$$

For flow along a rotating streamline:

Important for *fluid flows in turbomachinery*!

Compound (absolute) motion = relative motion + rotation (transfer motion).

$\Rightarrow$ Velocities

$\vec{w}$ - *relative velocity* - tangential to the rotating stream line (see *Fig. 3.19* and *Fig. 3.20*)!

$\vec{u}$ - *peripheral velocity* - normal to the radius of the rotating point (see *Fig. 3.19* and *Fig. 3.20*)!

$\vec{v}$ - *absolute velocity* - vector sum of $\vec{w}$ and $\vec{u}$ (see *Fig. 3.20*)!

$\therefore$

$$\vec{v} = \vec{w} + \vec{u} \quad (3\text{-}41)$$

$$\vec{u} = \left[\vec{\omega}, \vec{r}\right] = R\omega\vec{u_0} \quad (3\text{-}42)$$

$u = R\omega = r\omega \sin\alpha$ in $\frac{\text{m}}{\text{s}}$ - peripherial velocity intensity (see *Fig. 3.19*);

$\omega = 2\pi n$ s^{-1} - angular velocity; n - rotations/second.

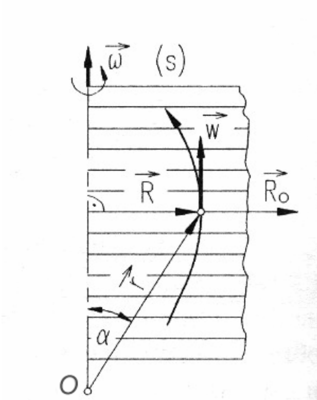


Fig. 3.19: Relative and peripherial velocity

$\omega = const$ = for *steady flow*!
$\Rightarrow$ *acceleration* of such absolute flow:

$$\vec{a} = \frac{d\vec{w}}{dt} + \vec{a}_c + \vec{a}_{ko} \tag{3-43}$$

$\frac{d\vec{w}}{dt}$ - *relative movement acceleration*; $\vec{a}_c$ - *centripetal acceleration*; $\vec{a}_{ko}$ - *Coriolis acceleration*.

$$\vec{a}_c = -R\omega^2\vec{R}_0 = -R\omega^2\left[\vec{\omega}_0, \vec{u}_0\right] = -\left[\omega\vec{\omega}_0, R\omega\vec{u}_0\right] = -\left[\vec{\omega}, \vec{u}\right] = -\left[\vec{\omega}, \left[\vec{\omega}, \vec{r}\right]\right] \tag{3-44}$$

$\vec{R}_0, \vec{u}_0, \vec{\omega}_0$ - *orts* (unit vectors) of the corresponding vectors.

$$\vec{a}_{ko} = 2\left[\vec{\omega}, \vec{w}\right] \tag{3-45}$$

$$\therefore \qquad \vec{a} = \frac{d\vec{w}}{dt} - \left[\vec{\omega}, \left[\vec{\omega}, \vec{r}\right]\right] + 2\left[\vec{\omega}, \vec{w}\right] \tag{3-46}$$

For 2-D flow with rotation axis normal to the flow plane $\Rightarrow \vec{w} \perp \vec{\omega}$; and $\vec{r} = \vec{R}$ (see *Fig. 3.20*):

$$\Rightarrow \qquad \left(\vec{\omega}, \vec{w}\right) = 0 ; \qquad \vec{a}_c = -R\omega^2\vec{R}_0 = -\omega^2\vec{R} = -\omega^2\vec{r} \tag{3-47}$$

The following scalar products are also zero:

$$\left(\vec{a}_{ko}, \vec{\omega}\right) = \left(\vec{a}_{ko}, \vec{w}\right) = \left(\vec{a}_{ko}, d\vec{s}\right) = 0 \tag{3-48}$$

$\therefore$ From the equation (3-46) *the overall acceleration* is $\Rightarrow$

$$\vec{a} = \frac{d\vec{w}}{dt} - \omega^2\vec{r} + 2\left[\vec{\omega}, \vec{w}\right] \tag{3-49}$$

The intensity of the *acceleration component tangential to the streamline* a_T (*Fig. 3.20*) is:

$$a_t = \left(\vec{a}, \vec{w}_0\right) = \frac{dw}{dt} - \omega^2 r\left(\vec{r}_0, \vec{w}_0\right) \tag{3-50}$$

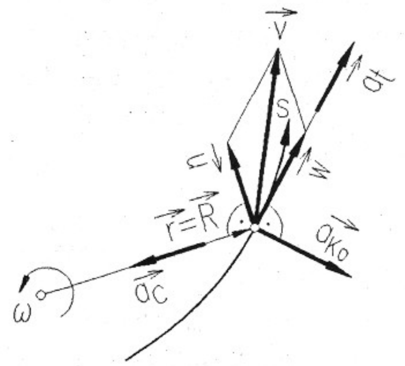


Fig. 3-20: Velocity and acceleration components of flow along a rotating streamline

4. Dynamics of inviscid (ideal) fluid flow

4.1. Forces on inviscid fluid flow, Euler equations for inviscid fluid flow

❖ ***Forces on inviscid fluid flow***

A fluid which has no resistance to shear stress is known as an ideal fluid or inviscid fluid.

∴ *The tangential surface forces are neglected; e.g.* $\tau = \mu \dfrac{dv}{dn} = 0$.

Acting forces on a fluid element with $dm = \rho dV$ - see also chapter 1.4 (*Fig. 1.2*) , and chapter 2.1 (*Fig. 2.1, Fig. 2.2 & Fig. 2.3*):

Inertial forces: $d\vec{F}_i = d\vec{J} = -dm\vec{a} = -dm\left(\dfrac{d\vec{v}}{dt}\right)$

Inertial force per unit mass ⇒ $\vec{J} = \dfrac{d\vec{J}}{dm} = -\dfrac{d\vec{v}}{dt} = -\vec{a}$ (4-1)

Surface forces - only normal pressure forces are acting (idial fluid)⇒ components:

$$dP_x = -\frac{\partial p}{\partial x}dxdydz = -\frac{\partial p}{\partial x}dV\ ;\quad dP_y = -\frac{\partial p}{\partial y}dV\ ;\quad dP_z = -\frac{\partial p}{\partial z}dV \qquad \text{(4- 2a)}$$

from *Fig. 2-3*: ⇒ $dP_y = (p_A - p_B)dxdz = -\dfrac{\partial p}{\partial y}dydxdz = -\dfrac{\partial p}{\partial y}dV$

⇒ *Surface forces - resultant:*

$$d\vec{P} = -\left(\frac{\partial p}{\partial x}\vec{i} + \frac{\partial p}{\partial y}\vec{j} + \frac{\partial p}{\partial z}\vec{k}\right)dV = -\text{grad}\, p\ dV = -\frac{dm}{\rho}\text{grad}\, p \qquad \text{(4-2)}$$

⇒ *Resultant surface forces per unit mass:*

$$\vec{P} = \frac{d\vec{P}}{dm} = -\frac{1}{\rho}\text{grad}\, p \qquad \text{(4-3)}$$

Elementary resultant body force $\vec{R}$ in N/kg (see equation (2-3) in chapter 2.1:

$$\vec{R} = X\vec{i} + Y\vec{j} + Z\vec{k} \qquad \text{(4-4)}$$

X,Y and Z - components of $\vec{R}$ in *x, y* and *z* directions.

According the *D'Alembert's principle* for dynamic equilibrium ⇒ :

$$\vec{J} + \vec{P} + \vec{R} = 0$$

i.e.

$$-\frac{d\vec{v}}{dt} - \frac{1}{\rho}\text{grad}\, p + \vec{R} = 0$$

∴ *Basic vector equation for inviscid (ideal) fluid flow:*

$$\frac{d\vec{v}}{dt} = \vec{R} - \frac{1}{\rho}\text{grad}\, p \qquad \text{(4-5)}$$

❖ ***Euler's equations for inviscid fluid flow, 3-D and 2-D flows***

The vector equation (4-5) can be expressed in *scalar form* ⇒

Components of body force $\vec{R} = \vec{R}(x, y, z)$ (chapter 2.1) - $X = \frac{\partial U}{\partial x}$; $Y = \frac{\partial U}{\partial y}$; $Z = \frac{\partial U}{\partial z}$ -

$$\Rightarrow \qquad \vec{R} = X\vec{i} + Y\vec{j} + Z\vec{k} = \frac{\partial U}{\partial x}\vec{i} + \frac{\partial U}{\partial y}\vec{j} + \frac{\partial U}{\partial z}\vec{k} = \mathrm{grad}\, U \qquad (4\text{-}7)$$

Components of the inertial forces per unit mass - see equation (4-1):

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\vec{i} + \frac{dv_y}{dt}\vec{j} + \frac{dv_z}{dt}\vec{k} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k} \qquad (4\text{-}8)$$

The acceleration component in the x direction - see equations (3-18):

$$a_x = \frac{dv_x}{dt} = \frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z} \qquad (4\text{-}9)$$

Similar expressions are for a_y and a_z - see equations (3-18) in the chapter 3.2.

The components of the surface forces can be defined through the pressure gradient - see equation (4-3):

$$\mathrm{grad}\, p = \frac{\partial p}{\partial x}\vec{i} + \frac{\partial p}{\partial y}\vec{j} + \frac{\partial p}{\partial z}\vec{k} \qquad (4\text{-}10)$$

∴ *The vector equation (4-5) can be transformed into three scalar equations:*

$$\frac{dv_x}{dt} = \frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z} = \frac{\partial U}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (4\text{-}11a)$$

$$\frac{dv_y}{dt} = \frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z} = \frac{\partial U}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad (4\text{-}11b)$$

$$\frac{dv_z}{dt} = \frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z} = \frac{\partial U}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} \qquad (4\text{-}11c)$$

∴ and together with the *continuity equation* - equation (3-28):

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = 0 \qquad (4\text{-}11d)$$

∴ This system of the 4 partial differential equations, (4-11a) to (4-11d), is known as *Euler's equations for 3-D inviscid (ideal) fluid flow.*

For *barotropic fluid*:

$$\rho = \rho(p); \qquad \text{for example } \frac{p}{\rho^\kappa} = const \qquad (4\text{-}11e)$$

∴ *The solution of the system of 5 governing equations* (4-11a) *to* (4-11e) *determines the components of the velocity, pressure and density:*

$$v_x = v_x(x,y,z,t); \quad v_y = v_y(x,y,z,t); \quad v_z = v_z(x,y,z,t); \quad p = p(x,y,z,t); \quad \rho = \rho(x,y,z,t)$$

However, in rare cases the analytical integration of the partial differential equations is possible!

For *steady flow* $\Rightarrow$ $\dfrac{\partial v_i}{\partial t}=0$ and $\dfrac{\partial \rho}{\partial t}=0$ (see also chapter 3.2)!

$\therefore$ For *steady 2-D flow*, $v_z=0$ and $\dfrac{\partial}{\partial z}=0$, $\Rightarrow$

$$v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}=\frac{\partial U}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{4-12a}$$

$$v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}=\frac{\partial U}{\partial y}-\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{4-12b}$$

$$\frac{\partial(\rho v_x)}{\partial x}+\frac{\partial(\rho v_y)}{\partial y}=0 \tag{4-12c}$$

For *incompressible fluid flow*, $\rho = const$, $\Rightarrow$

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0 \tag{4-12d}$$

4.2. One dimensional gravity flow - Bernnoulli's equation

❖ ***One dimensional gravity flow along a stream line* "s"** - see *Fig. 4.1a)* $\Rightarrow$

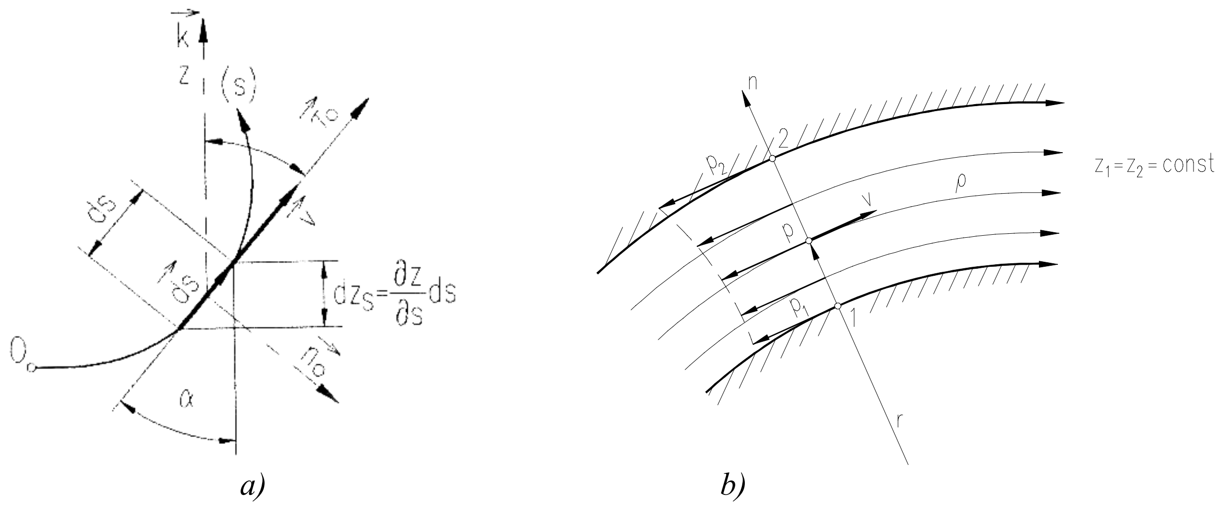
$$\frac{\partial U}{\partial x}=\frac{\partial U}{\partial y}=X=Y=0\,; \qquad Z=-g=\frac{\partial U}{\partial z}=\frac{dU}{dz}\,; \;\Rightarrow \qquad U=-gz+U_0 \tag{4-13}$$

If *z*-axis is opposite to the gravity force - *Fig. 4.1a)*, the resultant body (volume) force will be:

$$\vec{R}=Z\vec{k}=-g\vec{k}=\text{grad}U \tag{4-14}$$

$\therefore$ *The vector equation* (4-5) will transform to :

$$\frac{d\vec{v}}{dt}=-g\vec{k}-\frac{1}{\rho}\text{grad}\,p=\text{grad}U-\frac{1}{\rho}\text{grad}\,p \tag{4-15}$$



a) *b)*

Fig. 4.1: One dimensional gravity flow

This flow is convenient for forces equilibrium analysis along the streamline and normal to it.

❖ *Forces equilibrium along streamline s - tangential to s*:

Scalar product of the vector equation (4-15) and $d\vec{s}$ ⇒

$$\left(\frac{d\vec{v}}{dt}, d\vec{s}\right) = \left(\mathrm{grad}\,U, d\vec{s}\right) - \frac{1}{\rho}\left(\mathrm{grad}\,p, d\vec{s}\right) \qquad (4\text{-}16)$$

With the equations (3-39a and b) - chapter 3.5 (see also *Fig. 4.1a*) ⇒

$$\frac{d\vec{v}}{dt} = a_t\vec{\tau}_0 + a_n\vec{n}_0 = \left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial s}\right)\vec{\tau}_0 - \frac{v^2}{r_k}\vec{n}_0 \qquad (4\text{-}17)$$

The members of the equation (4-16) become:

$$\left(\frac{d\vec{v}}{dt}, d\vec{s}\right) = a_t ds = \left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial s}\right)ds \qquad (4\text{-}18a)$$

$$\left(\mathrm{grad}\,U, d\vec{s}\right) = dU_s = \frac{\partial U}{\partial s}ds = -g\frac{\partial z}{\partial s}ds = -g dz_s \qquad (4\text{-}18b)$$

$$\frac{1}{\rho}\left(\mathrm{grad}\,p, d\vec{s}\right) = \frac{1}{\rho}dp_s = \frac{1}{\rho}\frac{\partial p}{\partial s}ds \qquad (4\text{-}18c)$$

With the obtained expressions (4-18), the equation (4-16) is transformed to:

$$\frac{\partial v}{\partial t}ds + v\frac{\partial v}{\partial s}ds + g\frac{\partial z}{\partial s}ds + \frac{1}{\rho}\frac{\partial p}{\partial s}ds = 0 \ , \qquad (4\text{-}19)$$

wich, for barotropic fluid $\rho = \rho(p)$, can be writen as:

$$\frac{\partial v}{\partial t}ds + d\left(\frac{v^2}{2} + gz + \int\frac{dp}{\rho}\right)_s = 0 \qquad (4\text{-}19a)$$

∴ The integration of the equation (4-19a) along the streamline gives the *Bernoulli equation for unsteady inviscid compressible fluid flow along a streamline*:

$$\int_0^s \frac{\partial v}{\partial t}ds + \frac{v^2}{2} + gz + \int\frac{dp}{\rho} = const \qquad (4\text{-}20)$$

∴ Obviously the *Bernoulli equation for incompressible flow* ($\rho = const$) is:

$$\int_0^s \frac{\partial v}{\partial t}ds + \frac{v^2}{2} + gz + \frac{dp}{\rho} = const \qquad (4\text{-}21)$$

⇒ *Bernoulli equation for steady inviscid compressible fluid flow*:

$$\frac{v^2}{2} + gz + \int\frac{dp}{\rho} = const \qquad (4\text{-}22)$$

⇒ *Bernoulli equation for steady inviscid incompressible fluid flow*:

$$\frac{v^2}{2} + gz + \frac{p}{\rho} = const \qquad (4\text{-}23)$$

$\Rightarrow$ *the well-known Bernoulli equation for steady inviscid incompressible fluid flow (no energy losses)*:

$$\frac{v^2}{2g} + z + \frac{p}{\gamma} = const \qquad (4\text{-}24)$$

where: $\gamma = \rho g$

$\therefore$ *The Bernoulli equation is a form of the law for conservation of energy (see Fig. 4.2) !*

- *In the equation (4-23) every member presents a specific energy in* $\frac{\mathrm{Nm}}{\mathrm{kg}} = \frac{\mathrm{J}}{\mathrm{kg}}$;
- *In the equation (4-24) every member presents a specific energy in* $\frac{\mathrm{Nm}}{\mathrm{N}} = \mathrm{m}$

$\frac{v^2}{2g}$ - *kinetic energy*; z - *position or potential energy*; $\frac{p}{\gamma}$ - *pressure energy* (see *Fig. 4.2*).

$$\Rightarrow \qquad \frac{v_1^2}{2g} + z_1 + \frac{p_1}{\gamma} = \frac{v_2^2}{2g} + z_2 + \frac{p_2}{\gamma} = const \qquad (4\text{-}25)$$

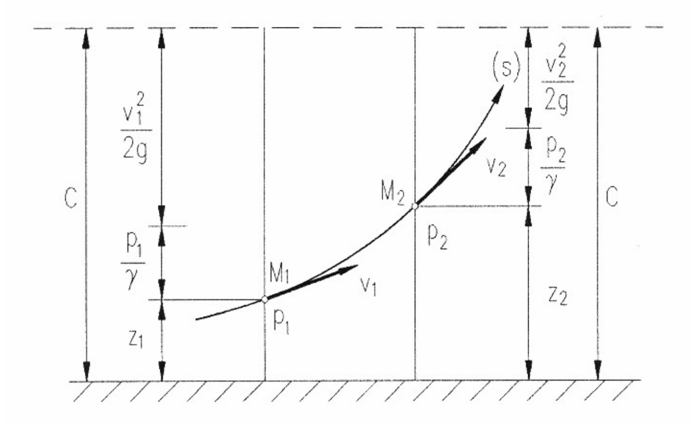


Fig. 4.2: Bernoulli equation as law for conservation of energy

❖ *Forces equilibrium along the normal "n" - see Fig. 4.1a* $\Rightarrow$:

- inertial force: $\left(\frac{d\vec{v}}{dt}, d\vec{n}\right) = a_n dn = -\frac{v^2}{r_k} dn$

- body (volume) forces: $-\left(g\vec{k}, d\vec{n}\right) = -g\frac{\partial z}{\partial n} dn = -g dz_n$

- pressure forces: $\left(\frac{1}{\rho}\operatorname{grad} p, d\vec{n}\right) = \frac{1}{\rho} dp_n = \frac{1}{\rho}\frac{\partial p}{\partial n} dn$

$$\therefore \qquad -\frac{v^2}{r_k} dn + \frac{1}{\rho}\frac{\partial p}{\partial n} dn + g\frac{\partial z}{\partial n} dn = 0 \qquad (4\text{-}26)$$

- *If the stream line is a straight line* ($r_k = \infty$), *for steady inviscid incompressible fluid flow* :

$$\Rightarrow \quad \frac{1}{\rho} dp_n + g dz_n = 0 \quad \Rightarrow \quad d_n\left(\frac{p}{\rho} + gz\right) = 0 \quad \Rightarrow \quad p + \gamma z = const \qquad (4\text{-}26a)$$

- *For a flow in a horizontal plane - Fig. 4.1b),* $dz_n = \frac{\partial z}{\partial n} dn = 0 \quad \Rightarrow$:

$$\Rightarrow \qquad dp_n = \rho \frac{v^2}{r_k} dn \qquad \Rightarrow \qquad \frac{dp_n}{dn} = \rho \frac{v^2}{r_k} \tag{4-26b}$$

- *If the stream line is a straight line* ($r_k = \infty$), $\Rightarrow \quad p = const$ *along the normal "n".*

❖ ***Flow along a rotating streamline*** - see *Fig. 4.4,* see also chapter 3.5 $\Rightarrow$

From the equation (3-49): $\vec{a} = \frac{d\vec{w}}{dt} - \omega^2 \vec{r} + 2[\vec{\omega}, \vec{w}]$,

and the general vector equation (4-5): $\frac{d\vec{v}}{dt} = \vec{R} - \frac{1}{\rho} \text{grad}\, p \quad \Rightarrow$

$$\frac{d\vec{w}}{dt} - \omega^2 \vec{r} + 2[\vec{\omega}, \vec{w}] = \vec{R} - \frac{1}{\rho} \text{grad}\, p \tag{4-27}$$

By multiplying the equation (4-27) with an elementary arc $d\vec{s} = \vec{w}dt = d\vec{r}$, for steady flow along an arbitrary stream line "*s*" $\Rightarrow$

$$wdw - \omega^2 rdr = (\vec{R}, d\vec{s}) - \frac{1}{\rho}(gradp, d\vec{s}) \tag{4-28}$$

Where are:

$\vec{w}$, $d\vec{w}$, $d\vec{s}$ and $d\vec{r}$ are collinear and $[\vec{\omega}, \vec{w}] \perp d\vec{s} \quad \Rightarrow \quad ([\vec{\omega}, \vec{w}], d\vec{s}) = 0$

$U = U(x, y, z)$ and $\vec{R} = gradU \quad \Rightarrow \quad (\vec{R}, d\vec{s}) = (gradU, d\vec{s}) = dU_s$ and also $(gradp, d\vec{s}) = dp_s$

$\therefore$ The equation (4-28) is transformed into: $d\left(\frac{w^2}{2}\right) - d\left(\frac{r\omega^2}{2}\right)^2 = dU - \frac{dp}{\rho}$

$\therefore$ After the intergration the following equation is obtained:

$$\frac{w^2}{2} - U + \int \frac{dp}{\rho} - \frac{u^2}{2} = const \tag{4-29}$$

where: $u = r\omega$

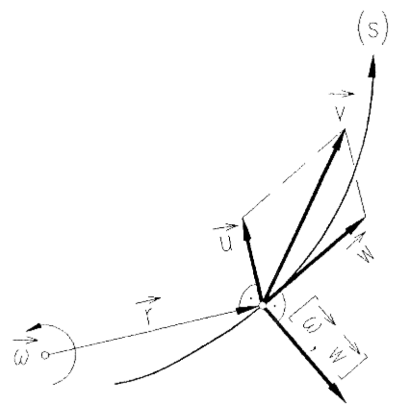


Fig 4.4: Flow along a rotating streamline

For gravity flow: $X = Y = \frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} = 0$; $Z = \frac{\partial U}{\partial z} = \frac{dU}{dz} = -g$; and $U = -gz + U_0 \Rightarrow$

$\therefore$ *The Bernoulli equation for compressible fluid flow along a rotating streamline:*

$$\frac{w^2}{2} + \int \frac{dp}{\rho} + gz - \frac{u^2}{2} = const \tag{4-30}$$

$\therefore$ *The Bernoulli equation for incompressible fluid flow ($\rho = const$) along a rotating streamline:*

$$\frac{p}{\rho g} + z + \frac{w^2}{2g} - \frac{u^2}{2g} = const \tag{4-31}$$

In the *Turbo-machinery* theory these equations can be applied for the entire flow field, and it is known as *Bernoulli equations for rotating channels.*

4.3. Potential flow - differential equations, Cauchy-Lagrange and Bernnoulli equation

*A **rotational fluid flow** can contain streamlines that loop back on themselves.*
Hence, fluid particles following such streamlines will travel along closed paths - vorteces.

*A **vortex** $\vec{\omega}$ in Fluid Mechanics is defined as:*

$$\vec{\omega} = \frac{1}{2}\mathrm{rot}\,\vec{v} = \frac{1}{2}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \omega_x \vec{i} + \omega_y \vec{j} + \omega_z \vec{k} \tag{4-32}$$

Bounded (and hence nonuniform) viscous fluids exhibit rotational flow, typically within their boundary layers. Since all real fluids are viscous to some amount, all real fluids exhibit a level of rotational flow somewhere in their domain.

*An **irrotational (potential) fluid flow** is one whose streamlines never loop back on themselves.*

Typically, only inviscid fluids can be irrotational. A uniform viscid fluid flow without boundaries is also irrotational, but this is a special (and boring!) case.

For a potential flow $\Rightarrow \vec{\omega} = 0 \Rightarrow$

$$\omega_x = \frac{1}{2}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) = 0, \quad \omega_y = \frac{1}{2}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) = 0, \quad \omega_z = \frac{1}{2}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) = 0 \tag{4-33}$$

It is obvious, from (4-33), that:

$$\frac{\partial v_z}{\partial y} = \frac{\partial v_y}{\partial z}, \quad \frac{\partial v_x}{\partial z} = \frac{\partial v_z}{\partial x}; \quad \frac{\partial v_y}{\partial x} = \frac{\partial v_x}{\partial y} \tag{4-34}$$

$\Rightarrow$ *Conclusion* from (4-34):

*A scalar **potential function** $\varphi = \varphi(x, y, z)$ can be defined for irrotational flow!*

$$\frac{\partial \varphi}{\partial x} = v_x, \quad \frac{\partial \varphi}{\partial y} = v_y, \quad \frac{\partial \varphi}{\partial z} = v_z \tag{4-35}$$

∴ *The velocity vector can be defined as:*

$$\vec{v} = \text{grad}\,\varphi = \frac{\partial \varphi}{\partial x}\vec{i} + \frac{\partial \varphi}{\partial y}\vec{j} + \frac{\partial \varphi}{\partial z}\vec{k} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k} \qquad (4\text{-}36)$$

According *the field theory* in mathematics, it is obvious from (4-36) that $v \perp \varphi$ (for 2-D flow, see *Fig. 4.5*).

∴ *The stream function* ψ (see equation (3-13) and *Fig. 4.5* for 2-D flow) *is also related to the potential function* φ ⇒*:*

$$v_x = \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}\ , \qquad v_y = \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x} \qquad (4\text{-}37)$$

∴ *The equipotential lines are normal to the stream lines,* $(\varphi = const) \perp (\psi = const)$.

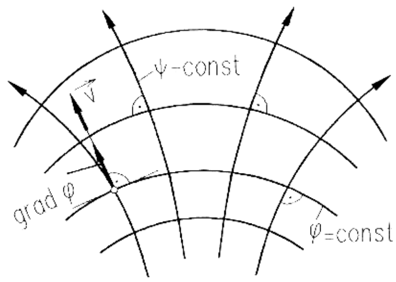


Fig 4.5: Stream lines versus equipotential lines

⇒ *The* ***Euler differential equations*** ((4-11a) to (4-11c)) *can be simplified* ***for potential flow****, by taking into account the above conclusion* - equations (4-34) and (4-35).

∴ *The equations* (4-11a) to (4-11c) *are transformed into:*

$$\begin{aligned} \frac{\partial v_x}{\partial t} + \frac{\partial}{\partial x}\left(P + \frac{v^2}{2} - U\right) &= 0 \\ \frac{\partial v_y}{\partial t} + \frac{\partial}{\partial y}\left(P + \frac{v^2}{2} - U\right) &= 0 \\ \frac{\partial v_z}{\partial t} + \frac{\partial}{\partial z}\left(P + \frac{v^2}{2} - U\right) &= 0 \end{aligned} \qquad (4\text{-}38)$$

Where $P = P(x,y,z) = \int \frac{dp}{\rho}$ is defined as "*generalized pressure*",

i. e. $\frac{\partial P}{\partial x} = \frac{1}{\rho}\frac{\partial p}{\partial x}$, $\frac{\partial P}{\partial y} = \frac{1}{\rho}\frac{\partial p}{\partial y}$, $\frac{\partial P}{\partial z} = \frac{1}{\rho}\frac{\partial p}{\partial z}$ in the *Euler differential equations* (4-11a-c) .

$U = U(x,y,z)$ - potential of the body force (chapter 2.1) - $X = \frac{\partial U}{\partial x}$; $Y = \frac{\partial U}{\partial y}$; $Z = \frac{\partial U}{\partial z}$.

Since from the equations (4-34) and (4-35) ⇒ $\frac{\partial v_x}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial t}\right)$, the system of the differential equations (4-38) is transformed into:

$$\frac{\partial}{\partial x}\left(P+\frac{\partial \varphi}{\partial t}+\frac{v^2}{2}-U\right)=0$$
$$\frac{\partial}{\partial y}\left(P+\frac{\partial \varphi}{\partial t}+\frac{v^2}{2}-U\right)=0 \qquad (4\text{-}39)$$
$$\frac{\partial}{\partial z}\left(P+\frac{\partial \varphi}{\partial t}+\frac{v^2}{2}-U\right)=0$$

∴ In the system (4-39) it is obvious that $P+\frac{\partial \varphi}{\partial t}+\frac{v^2}{2}-U=f(t)$ is function only of time (doesn't depend of *x,y* and *z*).

∴ *Finaly the **Cauchy-Lagrange** equation is obtained as a solution of the Euler's differential equations for a general case of potential compressible fluid flow:*

$$\int\frac{dp}{\rho}+\frac{\partial \varphi}{\partial t}+\frac{v^2}{2}-U=f(t) \qquad (4\text{-}40)$$

For steady flow, when $\partial\varphi/\partial t=0$, *the **Cauchy-Lagrange** equation is transformed into **Bernoulli equation** for steady compressible fluid flow* (see equation (4-22)):

$$\int\frac{dp}{\rho}+\frac{v^2}{2}-U=const \qquad (4\text{-}41)$$

4.4. Continiuty equation in integral form

The continuity equations in differential form were obtained in chapter 3.4 - see equations (3-28) to (3-30).

Consider an arbitrary control volume V_1 bounded by a surface $A_1=A_1^{'}+A_1^{''}$ (see *Fig. 4-16*).

The mass corresponding to corresponding to V_1 is:

$$m_1=\int_{V_1}\rho dV$$

$m_1=m_{V'}+m_{V_C}$; m_{V_C} - common volume (not shaded); $m_{V'}$ - belong to shaded part $V^{'}$

After time Δt, the fluid particles from the surface $A_1^{'}$ will make a path $\Delta s=v\Delta t$ and pass in the surface $A_2^{'}$ (entering the volume V_2). Also particles from $A_1^{''}\rightarrow A_2^{''}$.

∴ For steady flow, the fluid mass will pass in volume V_2 (bounded by $A_2=A_2^{'}+A_2^{''}$) and is defined as:

$$m_2=\int_{V_2}\rho dV$$

$m_2=m_{V''}+m_{V_C}+\Delta m$; $m_{V''}$ - belong to shaded part $V^{''}$

∴ For the time Δt, a mass difference Δm (belonnging to V_0 - *Fig. 4-16*), will be eventually created - "*source*" or "*sink*" ⇒:

$$\Delta m=\Delta m_0=m_2-m_1=\int_{V''}\rho dV-\int_{V'}\rho dV=d\int_{V}\rho dV$$

∴ *The time rate of mass flow change (elementary mass flow rate) will be*:

$$\frac{dm}{dt}=\frac{dm_0}{dt}=\frac{1}{dt}\left(\int_{V''}\rho dV-\int_{V'}\rho dV\right)=\frac{d}{dt}\int_{V}\rho dV \qquad \text{(4-42)}$$

From *Fig. 4.16*, an elementary volume dV can be defined as:

$$dV=dAds\cos\alpha=dAvdt\cos\alpha=\left(\vec{v},.d\vec{A}\right)dt$$

$ds=dh$ - according *Fig. 4.16*.

∴ The equation (4-42) can be transformed into *the continuity equation in integral form*:

$$\frac{dm}{dt}=\frac{d}{dt}\int_{V}\rho dV=\int_{A}\rho\left(\vec{v},d\vec{A}\right)=\int_{A}\rho dQ \qquad \text{(4-43)}$$

$\frac{dm}{dt}=\frac{dm_0}{dt}$ - mass flow rate from the volume V_0 through its bounding surface *O* (see *Fig. 4.16*).
$dQ=\left(\vec{v},d\vec{A}\right)$ - *elementary volume flow rate* (see equation (3-19).

⇒ The equation (4-43) can be transformed into:

$$\int_{K}\rho\left(\vec{v},d\vec{A}\right)=\int_{K}\rho dQ=0 \qquad \text{(4-44)}$$

$K=A+O$ - *closed control surface.*

If $V_0=0$ (there is no any "source" or "sink") $\Rightarrow \frac{dm}{dt}=0 \Rightarrow$:

∴ *The continuity equation in integral form for flow without singularities:*

$$\int_{A}\rho\left(\vec{v},d\vec{A}\right)=\int_{A}\rho dQ=0 \qquad \text{(4-45)}$$

∴ *The continuity equation in integral form for incompressible fluid flow ($\rho=const$ without singularities) will be:*

$$\int_{A}\left(\vec{v},d\vec{A}\right)=\int_{A}dQ=0 \qquad \text{(4-46)}$$

Compare the obtained continuity equations in integral form with the continuity equations in differential form obtained in chapter 3.4.

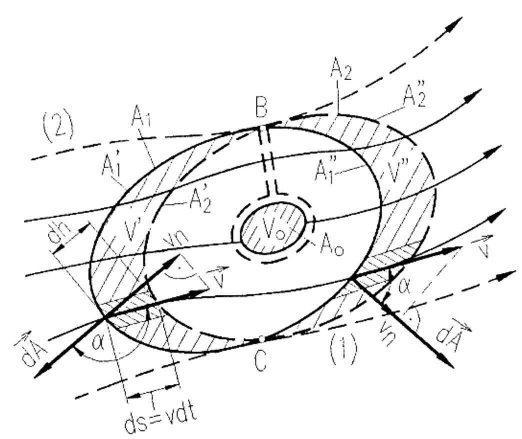


Fig. 4.16: Flow through an arbitrary finite control volume

4.5. Equations of momentum and energy

❖ ***Momentum and Moment of Momentum equations***

In Fluid Mechanics, it usually exists a flow of certain fluid quantity in space bounded by concrete surface - practically, there is no flow of ideal fluid particles.
⇒ A certain fluid mass *m* corresponds to a certain volume *V*. ⇒ The *Momentum (Impulse) law* from the *Solid Body Mechanics* (form the *II Newton's law*) can be applied:

$$\frac{d\vec{J}}{dt} = \vec{F}_R \tag{4-47}$$

$\vec{F}_R$ - *resultant force* acting on a certain mass *m*, causign its movement (see *Fig. 4.17*).
$\vec{J}$ - *Sum of the impulses* (*entire momentum*) of all elementary fluid particles.

For elementary fluid particle with $dm = \rho dV$, having $\vec{v} \Rightarrow d\vec{J} = dm\vec{v} = \rho\vec{v}dV$.

∴ *The entire momentum* will be:

$$\vec{J} = \int_V dm\vec{v} = \int_V \rho\vec{v}dV \tag{4-48}$$

∴ *The Momentum law in Fluid Mechanics* is defined as:

$$\frac{\overrightarrow{dJ}}{dt} = \frac{d}{dt}\int_V \rho\vec{v}dV = \vec{F}_R \tag{4-49}$$

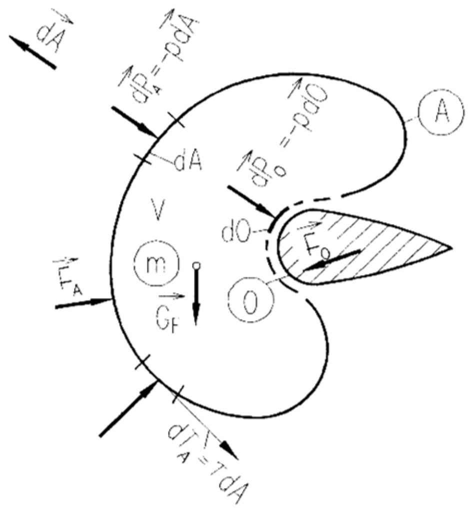


Fig. 4.17: Momentum concept for certain fluid mass m

Similarly, the *Moment of Momentum Law* can be defined as:

$$\frac{d\overrightarrow{M_F}}{dt} = \vec{M}_R \tag{4-50}$$

$\vec{M}_R$ - *Sum of the moments of all acting forces* on a certain mass *m*.
$\vec{M}_F$ - *Entire moment of momentum* - sum of moments for all elementary fluid particles.

The moment of momentum for certain fluid particle with mass *dm* and velocity $\vec{v}$ is:

$$\left[\vec{r}, dm\vec{v}\right] = dm\left[\vec{r}, \vec{v}\right]$$

$\vec{r}$ - distance of the fluid particle from the point to which the moment is considered.

∴ *The entire moment of momentum* will be:

$$\vec{M}_F = \int_V dm\left[\vec{r}, \vec{v}\right] = \int_V \rho\left[\vec{r}, \vec{v}\right] dV \tag{4-51}$$

∴ *The Moment of Momentum law in Fluid Mechanics* is defined as:

$$\frac{d\vec{M}_F}{dt} = \frac{d}{dt}\int_V \rho\left[\vec{r}, \vec{v}\right] dV = \vec{M}_R \tag{4-52}$$

Similarly to the concept of continuity equation derivation in the previous chapter 4.4, the *volume integral* can be transformed to *surface integral* (see equation (4-43)) ⇒

∴ *Momentum Law*:

$$\frac{d\vec{J}}{dt} = \int_K \rho\vec{v}dQ = \vec{F}_R \tag{4-53}$$

∴ *Moment of Momentum Law*:

$$\frac{d\vec{M}_F}{dt} = \int_K \rho\left[\vec{r}, \vec{v}\right] dQ = \vec{M}_R \tag{4-54}$$

$K = A + O$ - closed control surface bounding the mass m.

The *control surface* is recommended usually to be defined as on *Fig. 4.18* - around a possibly existing solid body with surface *O*:

- the boundary surfaces (A_3) are suggested to correspond to the streamlines - since, $dQ = 0$ on that part.
- the inlet (A_1) and outlet (A_2) surfaces are usually normal to the stream lines.

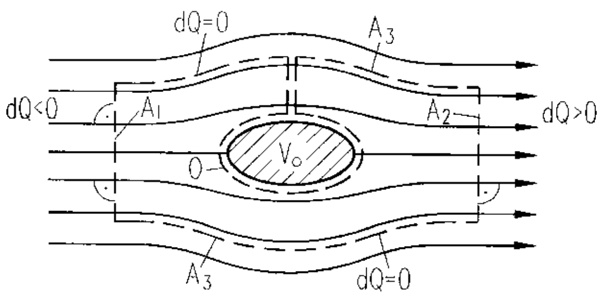


Fig. 4.18: Usual definition of a control surface

The resultant force acting on a certain fluid mass *m* is defined as (see *Fig. 4.19*):

$$\vec{F}_R = \vec{F}_A + \vec{F}_O + \vec{G}_F \tag{4-55}$$

$\vec{F}_A = \vec{P}_A + \vec{T}_A$ - *resultant surface force* acting on the surface *A*.

$\vec{P}_A = -\int_A p d\vec{A}$ - *normal surface forces* ; $\vec{T}_A$ - *tangential surface forces*.

$\vec{G}_F$ - *resultant body force* (gravity force, centrifugal force etc).

$\vec{F}_O = \vec{P}_O + \vec{T}_O$ - *resultant surface force* acting on the solid body surface *O*.

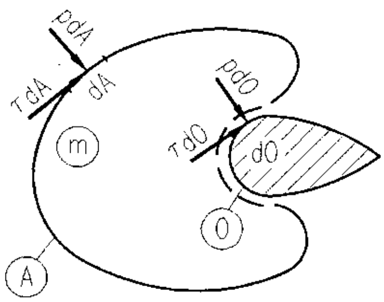


Fig. 4.19: Forces acting on fluid mass m and possible solid body

❖ ***General energy equation**:*

The first law of thermodynamics basically *states that a thermodynamic system can store or hold energy and that this internal energy is conserved.*

Heat is a process by which energy is added to a system from a high-temperature source, or lost to a low-temperature sink. In addition, energy may be lost by the system when it does *mechanical work* on its surroundings, or conversely, it may gain energy as a result of work done on it by its surroundings.

The first law states that this energy is conserved:

The change in the internal energy (du) is equal to the amount added by heating (dq) minus the amount lost by doing work on the environment (dw):

$$du = dq - dw \qquad (4\text{-}56)$$

du, *dq* and *dw* - specific energies (*energy per unit mass* expressed in Nm/kg or J/kg).

The equation (4-56) can be transformed into:

$$dq = du + d(p/\rho) + vdv + gdz \qquad (4\text{-}57)$$

Where:

$$dw = d(p/\rho) + vdv + gdz \qquad (4\text{-}58)$$

$d(p/\rho)$ - specific energy corresponding to *mechanical work of pressure*;

$vdv = d(v^2/2)$ - *specific kinetic energy*;

gz - *specific potential energy*.

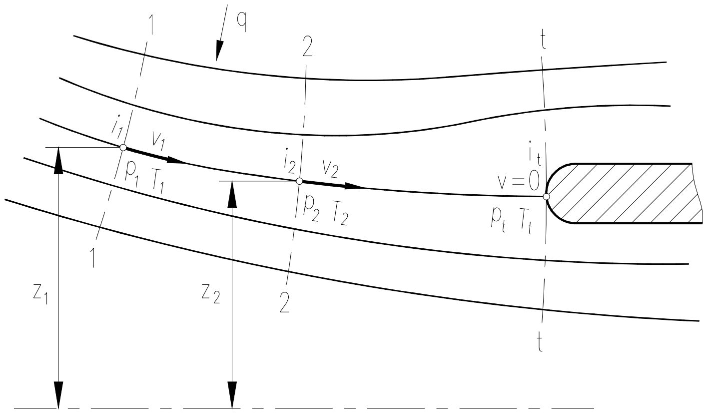


Fig. 4.20: First Law of thermodynamics - properties of fluid flow

Since, *specific enthalpy* is defined as:

$$i = u + \frac{p}{\rho} \tag{4-59}$$

$$\Rightarrow \qquad dq = di + vdv + gdz \tag{4-60}$$

For an *isentropic process* ($q_{1-2} = 0$) of flow between two flow sections1 and 2 (see *Fig. 4.20*), and flow in a hirizontal plane ($z_1 = z_2$), after the integration of the equation (4-60), the following equation is obtained:

$$i_1 - i_2 = \frac{v_2^2 - v_1^2}{2} = c_p\left(T_1 - T_2\right) \tag{4-61}$$

c_p - *specific heat at constant pressure* in J/kgK.

5. Some elementary flows of inviscid fluid

5.1. Stream tube control volume. Basic equations for flows through a stream tube

As defined in chapter 3.2 ⇒
A *stream tube* or *stream filament* is a small imaginary tube or "conduit" bounded by streamlines. Because the streamlines are tangent to the flow velocity, fluid that is inside a stream tube must remain forever within that same stream tube (see *Fig. 3.6* and *Fig. 5.1*).

∴ *The flow in a stream tube can be treated as flow in a pipe (no mixig with the surrounding).*

∴ *In many cases, like flow in pipes and channels, the concept of stream tube can be applied.*

∴ *Usualy the average flow properties are taken into account - at the central line of the stream tube, see Fig. 5.1.*

∴ *In practice, the flow through the stream tube can be treated as one-dimensional - see chapter 4.2.*

∴ **The basic equations for flow throw a stream tube** can be obtained (using the conclusions in previous chapter 4 and chapter 3) as follows:

- **Continuity equation**

The continuity equation in integral form (4-44) can be used ⇒:

$$\int_K \rho\left(\vec{v}, d\vec{A}\right) = \int_K \rho dQ = \int_{A_2} \rho dQ - \int_{A_1} \rho dQ = \int_{A_2} \rho\left(\vec{v}, d\vec{A}\right) - \int_{A_1} \rho\left(\vec{v}, d\vec{A}\right) = 0$$

$K = A_1 + A_2 + A_T + O$ - control section (see *Fig. 5.2*).

If $V_0 = 0$, and $O = 0$ (there is no any "source" or "sink" - *Fig. 5.1*) ⇒ $K = A_1 + A_2 + A_T$

Since through the boundaring surface A_T there is no inflow nor outflow,

$$\Rightarrow \qquad \int_{A_2} \rho\left(\vec{v}, d\vec{A}\right) = \int_{A_1} \rho\left(\vec{v}, d\vec{A}\right) = \int_{A} \rho\left(\vec{v}, d\vec{A}\right) \tag{5-1}$$

∴ *The mass flow rate is equal in every cross-section of the flow!*

Since the concept of average properties in certain cross-section (A) is applied (see *Fig. 5.1*)

$\Rightarrow (\vec{v}, d\vec{A}) = v dA \cos 0^0 = v dA \Rightarrow \int_A \rho\left(\vec{v}, d\vec{A}\right) = \int_A \rho v dA = \rho v \int_A dA = \rho v A = const \Rightarrow$

$$\rho_1 v_1 A_1 = \rho_2 v_2 A_2 = \rho v A = const \tag{5-2}$$

From (5-2) ⇒ $d\left(\rho v A\right) = 0$, and by dividing it with $\rho v A$ ⇒

$$\frac{d\rho}{\rho} = \frac{dv}{v} = \frac{dA}{A} = 0 \tag{5-3}$$

For incompressible fluid flow ($\rho = const$) ⇒

$$Q = v_1 A_1 = v_2 A_2 = vA = const \; ; \qquad \frac{dv}{v} = \frac{dA}{A} = 0 \tag{5-4}$$

For rotating channels $\Rightarrow$ $v = w$

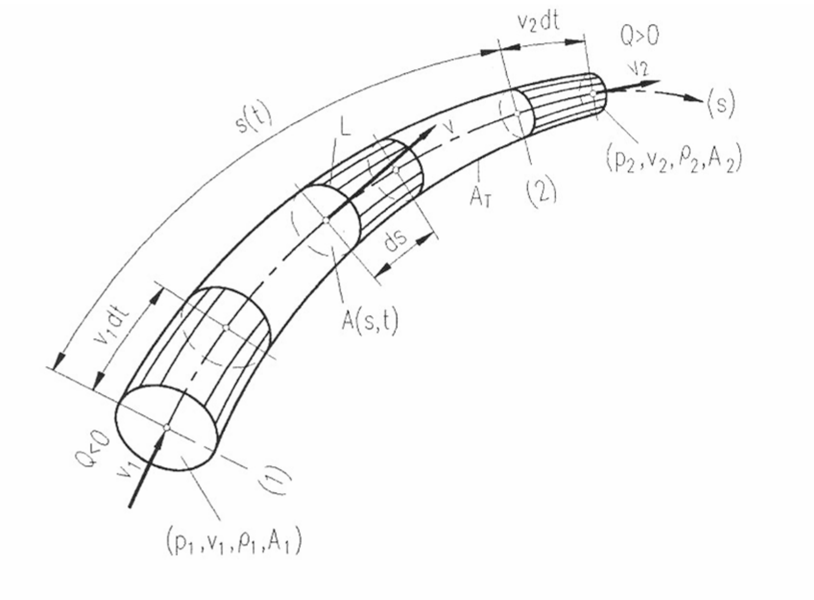


Fig. 5.1: Main properties of a stream tube

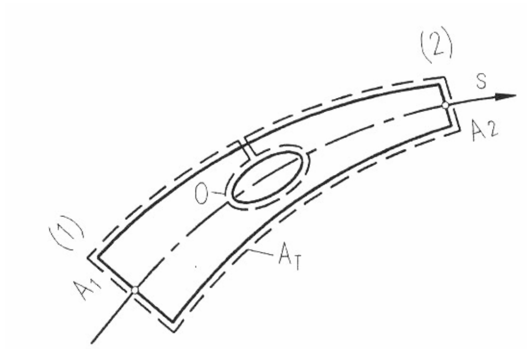


Fig. 5.2: Boundaring surfaces of a stream tube

- **Bernoulli's equation**

The equations for one-dimensional flow can be applied - see equations (4-20) to (4-25).

- *for steady inviscid compressible fluid flow*:

$$\frac{v^2}{2} + gz + \int \frac{dp}{\rho} = const \qquad (4\text{-}22)$$

- *for steady inviscid incompressible fluid flow*:

$$\frac{v^2}{2} + gz + \frac{p}{\rho} = const \qquad (4\text{-}23)$$

$$\Rightarrow \qquad \frac{v_1^2}{2g} + z_1 + \frac{p_1}{\gamma} = \frac{v_2^2}{2g} + z_2 + \frac{p_2}{\gamma} = const \qquad (4\text{-}25)$$

- **Momentum Law and Moment of Momentum Law**

From the equation (4-53) $\frac{\overrightarrow{dJ}}{dt} = \int_K \rho \vec{v} dQ = \vec{F}_R$, for control surface $K = A_1 + A_2 \Rightarrow$

$$\int_K \rho \vec{v} dQ = \int_{A_2} \rho \vec{v_2} dQ - \int_{A_1} \rho \vec{v_1} dQ = \vec{v_2} \int_{A_2} \rho dQ - \vec{v_1} \int_{A_1} \rho dQ$$

Taking into account the continuity equation (5-2) $\Rightarrow$

$$\int_K \rho \vec{v} dQ = \vec{v_2} \rho_2 Q_2 - \vec{v_1} \rho_1 Q_1 = \rho Q \left(\vec{v_2} - \vec{v_1} \right)$$

$\therefore$ *The Momentum Law for flow through stream tube* will be:

$$\rho Q \left(\vec{v_2} - \vec{v_1} \right) = \vec{F}_R \tag{5-5}$$

$\vec{F}_R$ - *Resultant force* acting on the fluid mass bounded by the control surface (see *Fig 5.3* and Fig. 5.4):

$$\vec{F}_R = \vec{P}_1 + \vec{P}_2 + \vec{F}_{1-2} + \vec{G}_{1-2} + \vec{P}_O$$

$\vec{P}_1 = -p_1 \vec{A}_1$; $\vec{P}_2 = -p_2 \vec{A}_2$; $\vec{F}_{1-2} = \vec{P}_{A_T} + \vec{T}_{A_T}$ - *surface forces* acting on the corresponding surfaces - see *Fig. 5.3*;
$\vec{G}_{1-2}$ - *body forces*;
$\vec{P}_O$ - a force acting from the surface of a possibly existing solid body inside the steam tube (see also *Fig. 4.18* in chapter 4.5).

If there is no a solid body $\Rightarrow$ $\vec{P}_O = 0$ $\Rightarrow$

$$\vec{F}_R = -p_1 \vec{A}_1 - p_2 \vec{A}_2 + \vec{F}_{1-2} + \vec{G}_{1-2}$$

$\therefore$ *The equation of Momentum Law for flow through stream tube* will be:

$$\rho Q \left(\vec{v_2} - \vec{v_1} \right) = -p_1 \vec{A}_1 - p_2 \vec{A}_2 + \vec{F}_{1-2} + \vec{G}_{1-2} \tag{5-6}$$

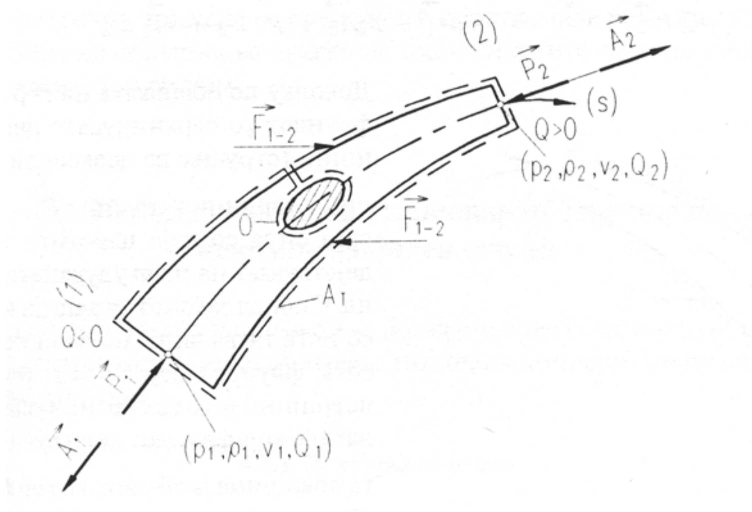


Fig. 5.3: Acting forces on a stream tube

If the stream tube has solid boundaries - like in pipes and channels (see *Fig. 5.4*), the acting force from the fluid to the solid boundaries, will be a reaction to the force $\vec{F}_{1-2} \Rightarrow$ $\vec{F}_r = -\vec{F}_{1-2} \Rightarrow$:

$$\vec{F}_r = \rho Q(\vec{v}_1 - \vec{v}_2) - p_1\vec{A}_1 - p_2\vec{A}_2 + \vec{G}_{1-2} \tag{5-7}$$

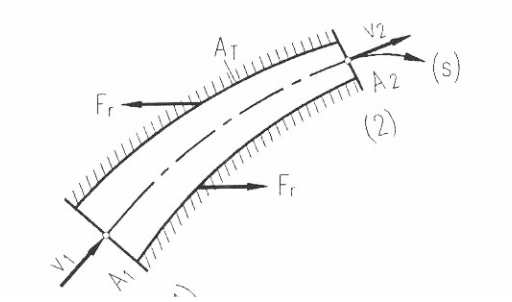


Fig. 5.4: Stream tube with solid boundaries

$\therefore$ *Moment of Momentum Law* can be derived in a similar manner using the previously obtained equation (4-54) in chapter 4.5.

However, an easier derivation can be performed by making vector products of every member of the equation (5-7) (i.e. every force) with the corresponding distance from the point to which the moment is considered (vector of its position) $\Rightarrow$

$$\vec{M}_r = \left[\vec{F}_r, \vec{r}_r\right] = \rho Q\left(\left[\vec{v}_1, \vec{r}_1\right] - \left[\vec{v}_2, \vec{r}_2\right]\right) - p_1\left[\vec{A}_1, \vec{r}_1\right] - p_2\left[\vec{A}_2, \vec{r}_2\right] + \left[\vec{G}_{1-2}, \vec{e}\right] \tag{5-8}$$

In practice, the vector equations (5-7) and (5-8) usually are interpreted equations in the directions of the chosen coordinate system.

5.2. Some examples of steady flow of incompressible fluid

- **Ventury pipe**

The concept of the Ventury tube is presented on Fig. 5.5.

For incompressible fluid flow ($\rho = const$) the derived equations in chapter 5.1 can be applied $\Rightarrow$

Continuity equation: $Q = v_1A_1 = v_2A_2 \quad \Rightarrow \quad v_2 = v_1A_1/A_2$

Bernoulli's equation: $\dfrac{v_1^2}{2g} + z_1 + \dfrac{p_1}{\rho g} = \dfrac{v_2^2}{2g} + z_2 + \dfrac{p_2}{\rho g}$

$\Rightarrow$ $\dfrac{p_1 - p_2}{\rho g} = \dfrac{1}{2g} v_1^2\left[(A_1/A_2)^2 - 1\right]$

$$v_1 = \sqrt{\frac{2g(p_1 - p_2)/\rho g}{(A_1/A_2)^2 - 1}} \tag{5-9}$$

For $\Delta h = h_1 - h_2 = (p_1 - p_2)/\rho g$, see Fig. 5.5, *the volume flow rate* is obtained as:

$$Q = v_1A_1 = A_1\sqrt{\frac{2g\Delta h}{(A_1/A_2)^2 - 1}} = C\sqrt{\Delta h} \tag{5-10}$$

where: $C = A_1\sqrt{\dfrac{2g}{(A_1/A_2)^2 - 1}}$

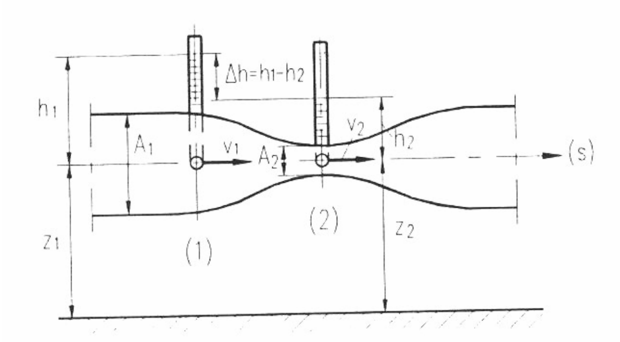


Fig. 5.5: Concept of Ventury tube (Ventury meter)

The Ventury pipe found an application as a device for *volume flow rate measurment - Ventury flow meter*.
However, for real fluid flow the viscosity (μ) effects have to be considered.

Due to the fluid field similarity, the above-presented approach for *Ventury tube* can be applied for the *orifice meter* as well (*Fig. 5.6*).
Applying a correction factor *k*, *the volume flow rate* can be obtained from the equation:

$$Q = kv_1A_1 = kA_1\sqrt{\frac{2g\Delta h}{(A_1/A_2)^2 - 1}} = kC\sqrt{\Delta h} \qquad (5\text{-}11)$$

The correction factor *k*, depends of the orifice geometry, fluid properties and flow regime - obtained experimentally.

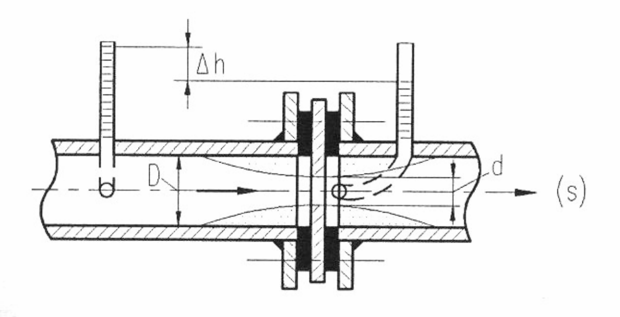


Fig. 5.6:Concept of an orifice meter co

- **Discharge from a reservoir into the atmosphere**

Discharge of incompressible inviscid fluid flow from a reservoir into the atmosphere is considered.

- Discharge through small nozzle - *Fig. 5.7*:

Bernoulli's equation between the free surface (0) and the nozzle outlet (1):

$$\frac{v_0^2}{2g} + z + \frac{p_0}{\rho g} = \frac{v^2}{2g} + \frac{p}{\rho g}$$

Continuity equation:

$$v_{01}A_0 = vA \qquad \Rightarrow \qquad v_0 = v\frac{A}{A_0}$$

Since $p = p_0 = p_a$ - atmospheric pressure $\Rightarrow$ $$\frac{v^2}{2g}\left(\frac{A}{A_0}\right)^2 + z = \frac{v^2}{2g} \qquad \Rightarrow$$

$$v = \sqrt{\frac{2gz}{1-(A_1/A_0)^2}} \tag{5-12}$$

Since, $A << A_0 \Rightarrow$

$$v = \sqrt{2gz} \tag{5-12a}$$

The equation (5-12a) is known as Torricelli's formula.

If the friction forces are taken into account, a correction factor $\varphi < 1$ has to be applied $\Rightarrow$:

$$v = \varphi\sqrt{2gz} \tag{5-13}$$

$\varphi = 0,96 \div 1,0$, obtained experimentally.

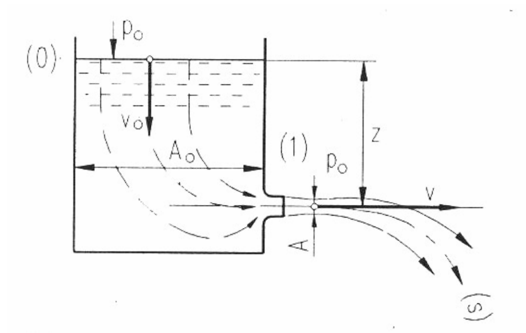


Fig. 5.7: Discharge through a small nozzle

The discharge through a nozzle is also accompanied by the jet contraction (see *Fig. 5.8*).
$\Rightarrow$ *a contraction factor*, $\psi = A^* / A < 1$ has to be taken into consideration $\Rightarrow$

$$Q = vA^* = \psi\varphi A\sqrt{2gz} = \mu A\sqrt{2gz} \tag{5-14}$$

$\mu = \psi\varphi < 1$ - *discharge coefficient*.

$0.5 < \mu < 1$ - experimentally obtained (see *Fig. 5.9*).

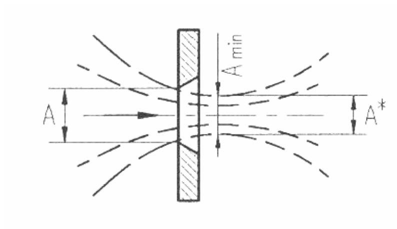


Fig. 5.8: Jet contraction

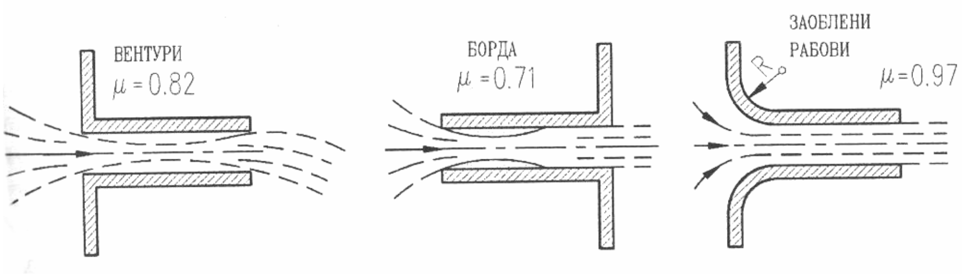


Fig. 5.9:Discharge coefficient for some nozzles

- Discharge into the atmosphere through large openings - *Fig. 5.10*:

For small nozzles in the *Torricelli's formula* (3-12a), $d << z \Rightarrow z \approx const$.

For large openings, *Fig. 5.10*, $\Rightarrow \; z \neq const$, $x = x(z)$, $\Rightarrow \; dA = x(z)dz$

$\Rightarrow$ *Elementary discharge*: $dQ = \mu v dA = \mu\sqrt{2gz}dA$

The entire discharge Q will be:

$$Q = \int_A dQ = \mu v dA = \mu\sqrt{2g}\int_{z=H_1}^{z=H_2} x(z)\sqrt{z}dz \qquad (5\text{-}15)$$

$\Rightarrow$ Example, *Discharge through a rectangular opening*:

$b = x(z) = const$; $H_2 - H_1 = a \;\Rightarrow$

$$Q = \mu\frac{2}{3}b\sqrt{2g}\left(\sqrt{H_2^3} - \sqrt{H_1^3}\right) \qquad (5\text{-}16)$$

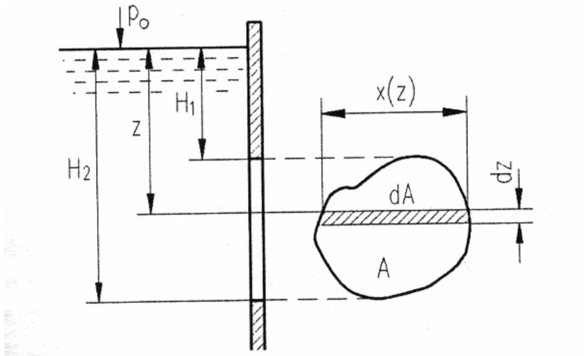


Fig. 5.10: Discharge into the atmosphere through large opening

$\Rightarrow$ Example, *Discharge through circular opening (Fig. 5.11)*:

$H_2 = H_1 + 2R$; $\left(\frac{x}{2}\right)^2 = R^2 - \left[(H_1 + R) - z\right]^2 \Rightarrow$ an elliptic integral:

$$Q = 2\mu\sqrt{2g}\int_{H_1}^{H_1+2R}\sqrt{z\left[R^2 - (H_1 + R - z)^2\right]}dz$$

If the under-integral function is developed into mathematical series $\Rightarrow$ the solution:

$$Q = \mu\pi R^2\sqrt{2g(H_1 + R)}\left[1 - \frac{1}{32}\frac{R^2}{(H_1 + R)^2}\right] \qquad (5\text{-}17)$$

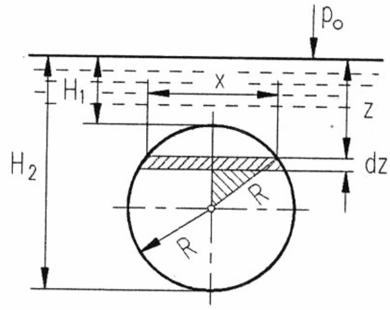


Fig. 5.11: Discharge into the atmosphere through circular opening

- **Submerged and partially submerged discharge through a large opening**

- ***Submerged discharge*** - *Fig. 5.12*:

$p_2 = p_0 + \gamma(z - H)$; $p_1 = p_0 + \gamma z$; $\Delta p = p_1 - p_2 = \gamma H = const$; $\Rightarrow$ $v = \sqrt{2gH} = const$

The discharge through the entire opening A will be:

$$Q = \mu A v = \mu A\sqrt{2gH} \qquad (5\text{-}18)$$

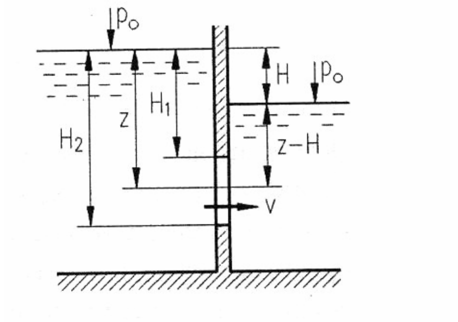


Fig. 5.12:Submerged discharge

- ***Partially submerged discharge*** - *Fig. 5.13*:

The entire discharge: $Q = Q_1 + Q_2$
Q_1 *- discharge into the atmosphere;* Q_2 *- submerged discharge.*
For rectangular opening: $A = (H_2 - H_1)b$*;* for Q_1 - equation (5-16); for Q_2 - equation (5-18).

$$\Rightarrow \qquad Q = \mu b\sqrt{2g}\left[\left(H_2 - \frac{H}{3}\right)\sqrt{H} - \frac{2}{3}H_1\sqrt{H_1}\right] \qquad (5\text{-}19)$$

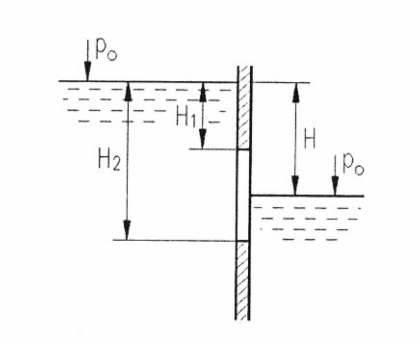


Fig. 5.13:Partially submerged discharge

- ***Discharge over a weir*** - *Fig. 5.14*:

Discharge over a weir through a rectangular opening is treated $\Rightarrow$ *a methodology for volume flow rate measurements* in open channels, rivers etc.

$H_1 = 0$; $H_2 = H$ $\qquad \Rightarrow \qquad Q = \mu\frac{2}{3}b\sqrt{2g}H\sqrt{H}$ (5-20)

Measurement section of H at $L = 3H$ - to avoid the overflow surface contraction.

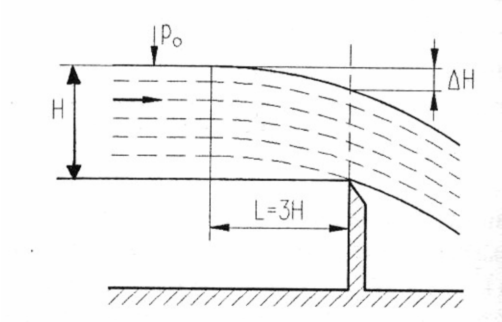


Fig. 5.14: Flow over a weir

- **A reservoir emptying time -** *Fig. 5.15*

from $Qdt = dV \quad \Rightarrow \quad \mu a\sqrt{2gz}dt = -A(z)dz$

$$\Rightarrow \qquad t = -\frac{1}{\mu a\sqrt{2g}}\int_{H_1}^{H_2}\frac{A(z)dz}{\sqrt{z}} = \frac{1}{\mu a\sqrt{2g}}\int_{H_2}^{H_1}\frac{A(z)dz}{\sqrt{z}} \qquad (5\text{-}21)$$

For prismatic reservoir, $A = const \;\Rightarrow$ $$t = \frac{A}{\mu a}\sqrt{\frac{2}{g}}\left(\sqrt{H_1} - \sqrt{H_2}\right) \qquad (5\text{-}21a)$$

The reservoir will be entirely emptied for $H_2 = 0 \;\Rightarrow$ $$t = \frac{A}{\mu a}\sqrt{\frac{2H_1}{g}} \qquad (5\text{-}21b)$$

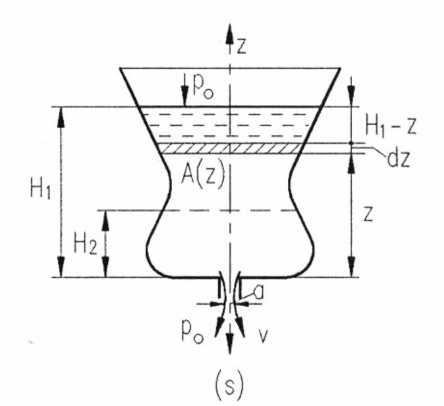


Fig. 5.15: Emptying of a reservoir

- **Flow through a rotating pipe, cavitation -** *Fig. 5.16*

Flow from a reservoir through a rotating pipe is considered (*Fig. 5.16*).

Bernoulli's equation from cross-section "*0*" to cross-section "*A*" - "*motionless (non-rotating) channel*":

$$\frac{v_0^2}{2g} + H_0 + \frac{p_0}{\rho g} = \frac{v_A^2}{2g} + h_A + \frac{p_A}{\rho g} \qquad (5\text{-}22)$$

Since, the reservoir cross-section >> pipe cross-section and $H_0 = const \;\Rightarrow\; v_0 = 0 \;\Rightarrow$

$$\frac{p_0 - p_A}{\rho g} = \frac{v_A^2}{2g} - (H_0 - h_A) \qquad (5\text{-}23)$$

According equation (4-31), the *Bernoulli's equation the rotating pipe* will be:

$$\frac{p_A}{\rho g}+h_A+\frac{w_A^2}{2g}-\frac{u_A^2}{2g}=\frac{p_2}{\rho g}+h_2+\frac{w_2^2}{2g}-\frac{u_2^2}{2g} \quad (5\text{-}24)$$

According *Fig. 5.16*: $w_A = v_A$ and $u_A = 0$; $p_2 = p_0$ and $h_2 = 0 \Rightarrow$

$$\frac{p_A}{\rho g}+h_A+\frac{v_A^2}{2g}=\frac{p_0}{\rho g}+\frac{w_2^2}{2g}-\frac{u_2^2}{2g} \quad (5\text{-}25)$$

Comparing the equations (5-22) and (5-25) $\Rightarrow$

$$\frac{p_0}{\rho g}+H_0+\frac{v_0^2}{2g}=\frac{p_0}{\rho g}+\frac{w_2^2}{2g}-\frac{u_2^2}{2g} \quad (5\text{-}26)$$

For pipe with constant cross-section $A_A = A_2$, from the continuity equation $Q = A_A w_A = A_2 w_2$ $\Rightarrow v_A = w_A = w_2$, and from the equation (5-25) $\Rightarrow$

$$\frac{p_0 - p_A}{\rho g} = h_A + \frac{u_2^2}{2g} > 0 \quad (5\text{-}27)$$

$$\frac{p_0}{\rho g} > \frac{p_A}{\rho g} \quad (5\text{-}28)$$

If $p_0 = p_{atmospheric} \Rightarrow p_A = vacuum$

If $p_A = p_k$ = *pressure of saturated vapor* $p_{s.v.}$ of the liquid at certain temperature $T \Rightarrow$ generation of vapor bubbles or cavities. These bubbles will implode at higher pressure $\Rightarrow$ *cavitation*!

p_k = *(liquid type, T) - example, for water at* $t = 20^0 C \Rightarrow p_k = p_{s.v.} = 0.0234$ bar $= 17.5$ mmHg.

The cavitation can also occur in non-rotational channels as well $\Rightarrow$ *explanation of Fig. 5.17.*

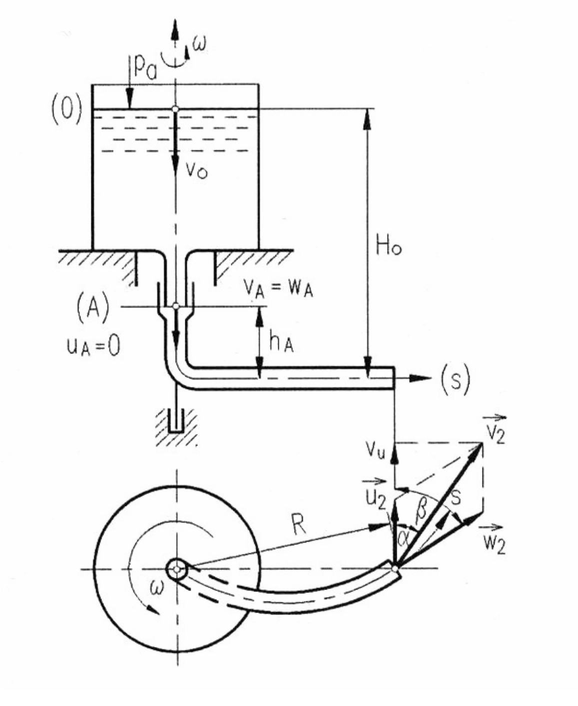


Fig. 5.16:Flow through a rotating pipe

Cavitation is a general term used to describe the behavior of *cavities* or bubbles in a liquid.

Cavitation is the process where generated liquid vapor bubbles rapidly (violently) collapse, producing shock waves - see Fig. 5.17. ⇒ *explanation*!

Cavitation may occur in pumps, propellers, impellers, and in the vascular tissues of plants.

∴ *The generation of liquid vapor bubbles (cavities) at low pressure, and their subsequent sudden implosion (violent closing) at higher pressure, under corresponding dinamic and static influences, is known as cavitation.*

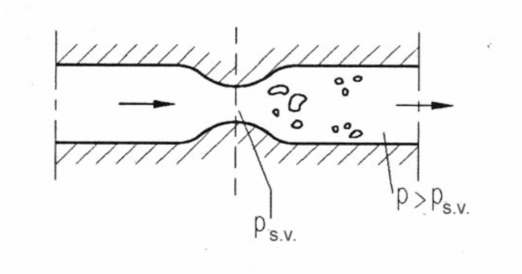


Fig. 5.17: Generation of cavitation in non-rotational channel

∴ *Cavitation is, in many cases, an undesirable occurrence. In devices such as propellers, pumps, and turbines, cavitation causes a great deal of noise, damage to components, vibrations, and a loss of efficiency - see Fig. 5.17a.*

Fig. 5.17a: Cavitation generation in a propeller (a) and cavitation damage of a turbine (b).

5.3. Basic consideration of compressible flow

$\rho = f(p, T, c_p, k, ...,)$ - for compressible fluid.

In general the density change can be determined from the energy equation (4-60):

$$dq = di + vdv + gdz$$

In the theory of heat and mass transfer, the following differential equation can be derived:

$$\rho c_p \frac{DT}{Dt} - \frac{Dp}{Dt} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \Phi \qquad (5\text{-}29)$$

k - heat conductivity coefficient; c_p *- specific heat*; $\Phi = (\lambda + \frac{2}{3}\mu)\theta^2$ *- dissipation function*;

$(\lambda + \frac{2}{3}\mu)$ *- volumetric viscousity;* $\theta = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$; $\frac{D}{Dt} = \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}$.

However, here, as an example a barotpropic fluid is treated:

$\Rightarrow$ $p = p(\rho)$

$\Rightarrow$ *Continuity equation for steady fluid flow* (5-2) $\Rightarrow$ $\rho_1 v_1 A_1 = \rho_2 v_2 A_2 = \rho v A = const$

$\Rightarrow$ *Bernoulli's equation for steady fluid flow* (4-22) $\Rightarrow$ $\frac{v^2}{2} + gz + \int \frac{dp}{\rho} = const$

$\therefore$ *v, p* and ρ can be defined from these last three equations.

- Some equations for adiabatic (isentropic) fluid flow:

Bernoulli's equation

See chapter 1.3 for gas properties and states.

For an adiabatic process, see equation (1-15) $\Rightarrow$ $\frac{p}{\rho^\kappa} = const$; $\kappa = \frac{c_p}{c_v}$

$$\Rightarrow \quad \frac{p}{\rho^\kappa} = \frac{p_0}{\rho_0^\kappa} = C \Rightarrow \rho = \left(\frac{p}{C}\right)^{\frac{1}{\kappa}} = \rho_0 \left(\frac{p}{p_0}\right)^{\frac{1}{\kappa}} \tag{5-30}$$

p_0, ρ_0 - *initial properties* (see *Fig. 5.22*).

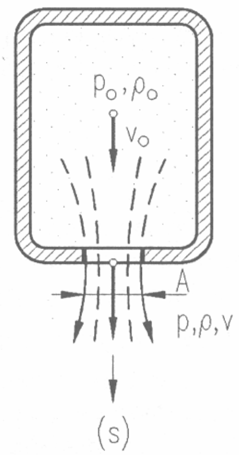


Fig. 5.22: Discharge of compressible fluid

From the *Bernoulli's equation* (4-22) $\Rightarrow$

$$\int \frac{dp}{\rho} = \int \left(\frac{p}{C}\right)^{-\frac{1}{\kappa}} dp = \frac{\kappa}{\kappa - 1} C^{\frac{1}{\kappa}} p^{\frac{\kappa-1}{\kappa}} = \frac{\kappa}{\kappa - 1} \frac{p_0^{\frac{1}{\kappa}}}{\rho_0} p^{\frac{\kappa-1}{\kappa}} \tag{5-31}$$

With the equation (5-30) $\Rightarrow$ the *Bernoulli's equation for steady adiabatic fluid flow*:
(for $z = 0$)

$$\frac{\kappa}{\kappa - 1} \frac{p}{\rho} + \frac{v^2}{2} = const \tag{5-32}$$

$$\frac{\kappa}{\kappa - 1} \frac{p}{\rho} + \frac{v^2}{2} = \frac{\kappa}{\kappa - 1} \frac{p_0}{\rho_0} + \frac{v_0^2}{2} \tag{5-33}$$

From chapter 1.3, equation (1-10) $\Rightarrow$ *velocity of sound* $c=\sqrt{\kappa p/\rho}$ $\Rightarrow$

$$\frac{c^2}{\kappa-1}+\frac{v^2}{2}=\frac{c_0^2}{\kappa-1}+\frac{v_0^2}{2} \tag{5-34}$$

Dischargeof compressible fluid through nozzles

Discharge of *adiabatic compressible fluid flow of ideal gas* is treated - as on *Fig. 5.22*.

The discharge velocity, from the equation (5-33), for $v_0 \approx 0$, $\Rightarrow$ *the Saint-Venant formula*:

$$v=\sqrt{2\frac{\kappa}{\kappa-1}\left(\frac{p_0}{\rho_0}-\frac{p}{\rho}\right)} \tag{5-35}$$

If the equation of state for ideal gas (1-11), $\frac{p}{\rho}=RT$, is taken into account (see chapter 1.3) $\Rightarrow$

$$v=\sqrt{2\frac{\kappa}{\kappa-1}R(T_0-T)} \tag{5-36}$$

Consider the equation (5-34) $\Rightarrow$

$$v=\sqrt{\frac{2}{\kappa-1}\left(c_0^2-c^2\right)} \tag{5-37}$$

For real gases and vapors $\Rightarrow$ use of tables and graphical curves obtained mostly experimentally $\Rightarrow$ empirical formula.

Example: flow through Ventury meter or an orifice meter - see *Fig. 5.5 and Fig. 5.6* and equation (5-11) $\Rightarrow$

$$\dot{m}=\varepsilon C_d A\sqrt{2\rho_1(p_1-p_2)};\quad Q=\frac{\dot{m}}{\rho_1}=\varepsilon C_d A\sqrt{\frac{2(p_1-p_2)}{\rho_1}}$$

$C_d = f_1(d/D, Re) = f(m, Re)$ - *discharge coefficient* - experimentally obtained;

$\varepsilon=\left(f\frac{d}{D}, p_1, (p_1-p_2), \kappa\right)$ - *coefficient of expansion* - experimentally obtained.

5.4. Some examples for the momentum equation application

- Force on bended pipe - *Fig. 5.24*:

From the derived equation (5-7), $\vec{F}_r=\rho Q(\vec{v}_1-\vec{v}_2)-p_1\vec{A}_1-p_2\vec{A}_2+\vec{G}_{1-2}$,

and the continuity equation $Q=v_1A_1=v_2A_2$, according *Fig. 5.24* $\Rightarrow$ $\vec{v}_1A_1=-v_1\vec{A}_1$ and $\vec{v}_2A_2=+v_2\vec{A}_2$ $\Rightarrow$

$$\vec{F}_r=-\left(p_1+\rho v_1^2\right)\vec{A}_1-\left(p_2+\rho v_2^2\right)\vec{A}_2+\vec{G}_{1-2} \tag{5-38}$$

The components of this force will be (see *Fig. 5.24*):

$$F_{rx}=\left(\vec{F}_r,\vec{i}\right)=\left(p_1+\rho v_1^2\right)A_1-\left(p_2+\rho v_2^2\right)A_2\cos\alpha\ ;\qquad F_{rz}=F_{rx}tg\beta \tag{5-39}$$

In this case, the corresponding scalar products are:
$\left(\vec{A}_1,\vec{i}\right)=A_1\cos\pi=-A_1$; $\left(\vec{A}_2,\vec{i}\right)=A_2\cos\alpha$; $\left(\vec{G}_{1-2},\vec{i}\right)=A_2\cos(\pi/2)=0$

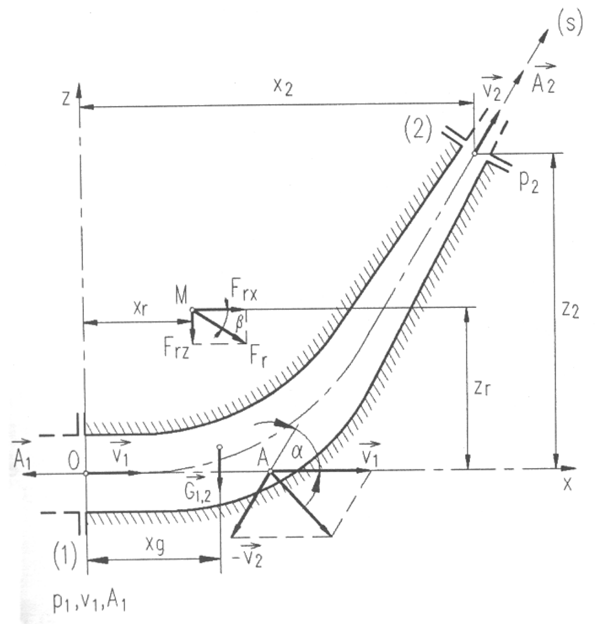


Fig. 5.24: Force on a bended pipe

- Jet reaction - *Fig. 5.25*:

Discharge into the atmosphere is treated - *Fig. 5.25* ⇒ the equation (5-12a) is valid ⇒

$$v = \sqrt{2gH}$$

∴ The equation (5-7), according *Fig. 5.25*, is transformed into:

$$\vec{F}_r = \rho Q(\vec{v}_0 - \vec{v}) - p_0 \vec{A}_0 - p_0 \vec{A} + \vec{G}$$

The resultant acting force (component) in the "z" direction will be:

$$F_{rz} = -F_{rz}^{'} - p_0 A_0 = -(\rho Q v_0 - p_0 A_0) - G - p_0 A_0 = -(\rho v_0^2 A_0 + G) \qquad (5\text{-}40)$$

where:

$p_0 A_0$ is also acting on the bottom from outside; $Q = v_0 A_0$

Adequately, the resultant acting force (component) in the "x" direction will be:

$$F_{rx} = -F_{rx}^{'} + p_0 A = -(p_0 + \rho v^2)A + p_0 A = -\rho v^2 A = -\rho Q v \qquad (5\text{-}41)$$

∴ *The force F_{rx} is known as the reaction to the jet! The sign "-" means that F_{rx} is in the opposite direction to the velocity of the discharge (see Fig. 5.25).*

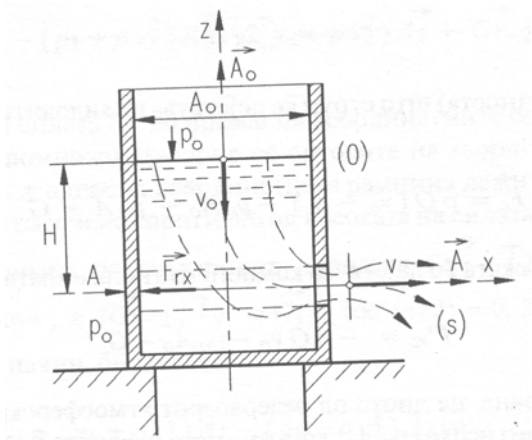


Fig. 5.25: Jet reaction

- **Missile reaction force and speed -** *Fig. 5.26*:

w - relative velocity of the gases through the Laval nozzle; $v = w - v_R$ *- absolute gas velocity;* v_R *- missile velocity.* $\Rightarrow$ from (5-7) $\Rightarrow$

$$F_r = -\rho Q w = -\rho A w^2 \tag{5-42}$$

From the II Newton's law $\Rightarrow$

$$F_r = -M\frac{dv_R}{dt} \tag{5-43}$$

$M = M_R + M_F - \rho Qt$ *- the missile mass after certain time "t".*
M_R *- the mass of the useful missile load;* M_F *- the fuel mass before the missile start;*
ρQt *- mass of the burned up gasses.*

$\therefore$ From the equations (5-42) and (5-43) $\Rightarrow$ *the missile velocity*:

$$-\frac{dv_R}{dt} = \frac{F_r}{M} = -\frac{\rho Q w}{M_R + M_F - \rho Qt} \tag{5-44}$$

Since, $Q = const$ and $w = const$ $\Rightarrow$

$$v_R = w\int_0^t \frac{\rho Q dt}{M_R + M_F - \rho Qt} = w\ln\left[1 + \frac{\rho Qt}{M_R + M_F - \rho Qt}\right] \tag{5-45}$$

After time $T = M_F / \rho Q$, *the maximum missile velocity* will be achieved:

$$v_{R\max} = w\ln\left[1 + \frac{M_F}{M_R}\right] \tag{5-46}$$

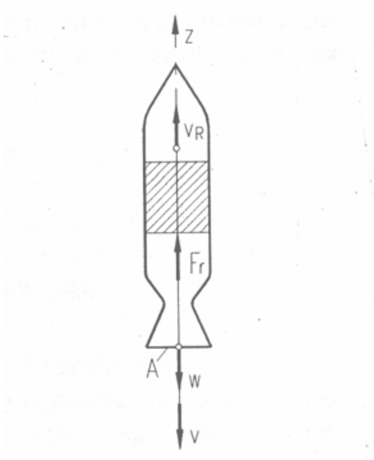


Fig. 5-26:Missile reaction force and speed

- **Basic equation of the turbo machines -** *Fig. 5.27*:

A steady flow of an inviscid fluid in a turbomachine runner is treated.

From the *Moment of Momentum*, derived equation (5-8)

$\vec{M}_r = \left[\vec{F}_r, \vec{r}_r\right] = \rho Q\left(\left[\vec{v}_1, \vec{r}_1\right] - \left[\vec{v}_2, \vec{r}_2\right]\right) - p_1\left[\vec{A}_1, \vec{r}_1\right] - p_2\left[\vec{A}_2, \vec{r}_2\right] + \left[\vec{G}_{1-2}, \vec{e}\right]$, for conditions when:

- the flow is in a horisontal plane G_{1-2} || to the axis of rotation $\Rightarrow \left[\vec{G}_{1-2}, \vec{e}\right] = 0$,
- and for cylindrical entrance and exit surfaces $\Rightarrow$ $\left[\vec{A}_i, \vec{r}_i\right] = 0$, $\Rightarrow$

⇒ *The turbomachine runner torque* will be:

$$\vec{M}_r = \rho Q\left(\left[\vec{v}_1, \vec{r}_1\right] - \left[\vec{v}_2, \vec{r}_2\right]\right) \tag{5-47}$$

Since, for collinear vectors, the intensities of the vector products are:

$$\left|\left[\vec{v}_i, \vec{r}_i\right]\right| = v_i r_i \sin\left(\frac{\pi}{2} - \alpha_i\right) = v_i r_i \cos\alpha_i \;\Rightarrow$$

$$\left|\vec{M}_r\right| = M_r = \rho Q\left(v_1 r_1 \cos\alpha_1 - v_2 r_2 \cos\alpha_2\right) = \rho Q\left(r_1 v_{1u} - r_2 v_{2u}\right) \tag{5-48}$$

where, according *Fig. 5.27*, $v_u = v\cos\alpha$ - see *Fig. 5.27b)*.

∴ *The equation (5-48) is the fundamental equation of turbomachines.*

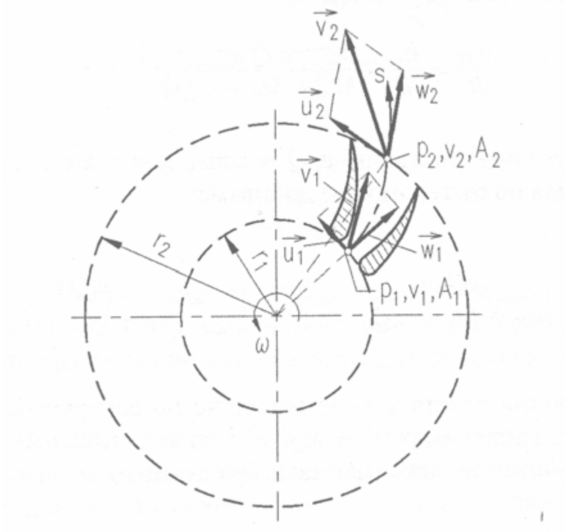
∴ *The corresponding power to this theoretical torque (inviscid fluid) will be:*

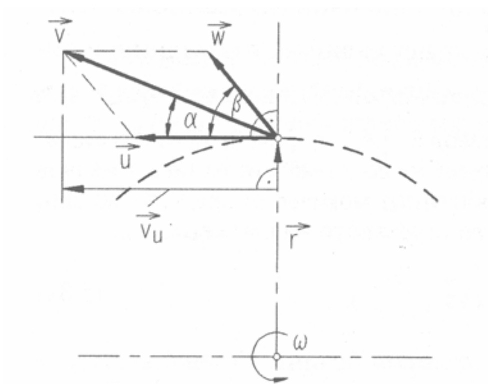
$$N = M_r\omega = \rho Q\left(r_1\omega v_{1u} - r_2\omega v_{2u}\right) = \rho Q\left(u_1 v_{1u} - u_2 v_{2u}\right) \tag{5-49}$$

From *Fig. 5.27b)* ⇒ $w^2 = u^2 + v^2 - 2vu\cos\alpha \;\Rightarrow$

$$N = \rho Q\left(\frac{u_1^2 - u_2^2}{2} + \frac{v_1^2 - v_2^2}{2} - \frac{w_1^2 - w_2^2}{2}\right), \tag{5-50}$$

∴ *N is theoretical power ⇒ energy losses have to be taken into account.*
∴ $N > 0$ *for a turbine - the energy is delivered to the axis of rotation (runner shaft).*
∴ $N < 0$ *for a pump - the energy is delivered from the axis of rotation (impeller shaft)*





a) b)

Fig. 5.27: Rotating channel of a turbomachine

6. Some fundamental concepts of viscous fluid flow

6.1. General concept of viscous fluid flow

All real fluids have some resistance to stress!
Viscosity is a measure of the resistance of a fluid to deform under shear stress.
$\Rightarrow$

$\therefore$ ***The Newton's law*** *for shear stress is valid:* $$\tau = \mu \frac{dv}{dn},\qquad (6\text{-}1)$$

- see equation (1-6) and *Fig. 1.1* in chapter 1.3

τ - *shear stress* in N/m^2; dv/dn - *rate of angular deformation* (velocity gradient) in s^{-1};

μ - *Dynamic (absolute) viscosity* in kg/ms=Ns/m^2.

$\Rightarrow \quad \nu = \dfrac{\mu}{\rho}$ - *Kinematic viscosity* in m^2/s.

Viscosity describes a fluid's internal resistance to flow and may be thought of as a measure of fluid friction.

$\therefore$ *The influence of tangential surface shear force has to be taken into account for real fluids.*

The influence of the viscosity, i.e. the shear stress has especially effect at the boundaries of a solid body $\Rightarrow$ change of the velocity profile, see *Fig. 6.1*.

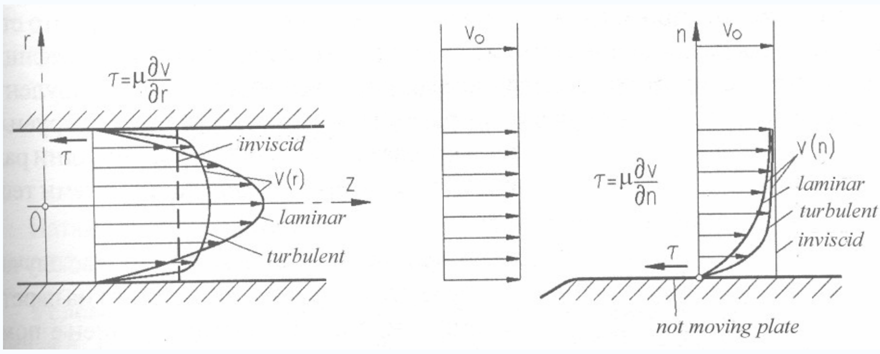


Fig. 6.1: The influence of the shear stress for viscous fluid flow

$\therefore$ **Flow classification** *can be defined:*

- *inviscid (ideal) fluid flow* - see chapter 4.;
- *viscous (real) fluid flow* - basic concepts in this chapter 6.

Depending on the relative magnitudes of the viscous and inertia forces two modes of viscous fluid flow can be defined:

- *laminar flow;*
- *turbulent flow*

Laminar flow, sometimes known as streamline flow, occurs when a fluid flows in parallel layers, with no disruption between the layers.

Laminar flow - an organized flow field that can be described with streamlines. In order for laminar flow to be permissible, the viscous stresses must dominate over the fluid inertia stresses.

Turbulent flow is a flow regime characterized by chaotic, stochastic property changes.

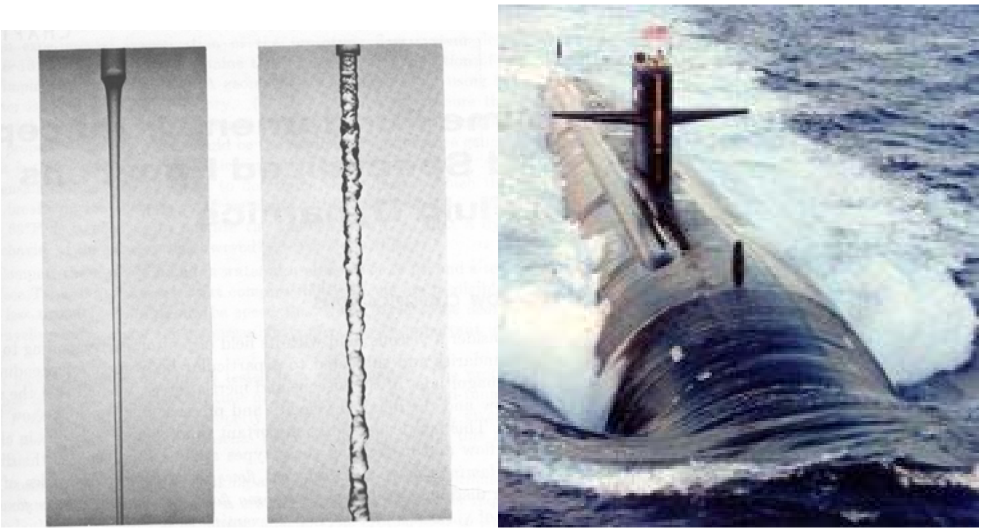
Turbulent flow - a flow field that cannot be described with streamlines in the absolute sense. However, time-averaged streamlines can be defined to describe the average behavior of the flow. In turbulent flow, the inertia stresses dominate over the viscous stresses, leading to small-scale chaotic behavior in the fluid motion.

$\therefore$ The dimensionless *Reynolds number* - Re (see later chapter 6.6, and chapter 7) is an important parameter in the equations that describe whether flow conditions lead to laminar or turbulent flow:

$$\frac{\text{inertia force/mass}}{\text{frictional force/mass}} \propto \text{Reynolds number}$$

$\mathrm{Re} < \mathrm{Re}_{cr}$ - *laminar flow*; $\mathrm{Re} > \mathrm{Re}_{cr}$ - *turbulent flow*.

Re_{cr} - *critical Reynolds number*, defined later in chapter 6.6.



a) laminar *b) turbulent* *c) Laminar and turbulent water flow over the hull of a submarine*

Fig. 6.2:Examples of laminar and turbulent flow

6.2. Fundamental equations for laminar flow

- *The continuity equation is valid* - equation (3-28):

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \vec{v}) = 0 \qquad \text{i.e.} \qquad \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = 0 \qquad (6\text{-}2)$$

- *The energy equation is also valid* - equation (4-60), $\Rightarrow$ the density change can be obtained:

$$\rho = f(p, T, c_p, k, ...,)$$

for barotropic fluid $\Rightarrow$

$$\rho = f(p) \qquad (6\text{-}3)$$

- *Euler's equations,* (4-11a) to (4-11c)*, can be also applied if tangential or shear stresses are taken into account* $\Rightarrow$ *Navier-Stockes equations!*

- **Stresses in a viscous fluid flow:**
- *Stress, is a measure of the average amount of force exerted per unit area - e.g.* $\sigma = \frac{P}{A}$.

$\therefore$ *The influence of tangential surface shear force has to be taken into account for real fluids,* besides the forces defined for ideal fluid (see chapter 4.1)

- Infinitesimal fluid element at point *M(x,y,z)* is considered in the fluid flow

$\Rightarrow dV = dxdydz$; $dm = \rho dV$ - see *Fig. 6.3*.

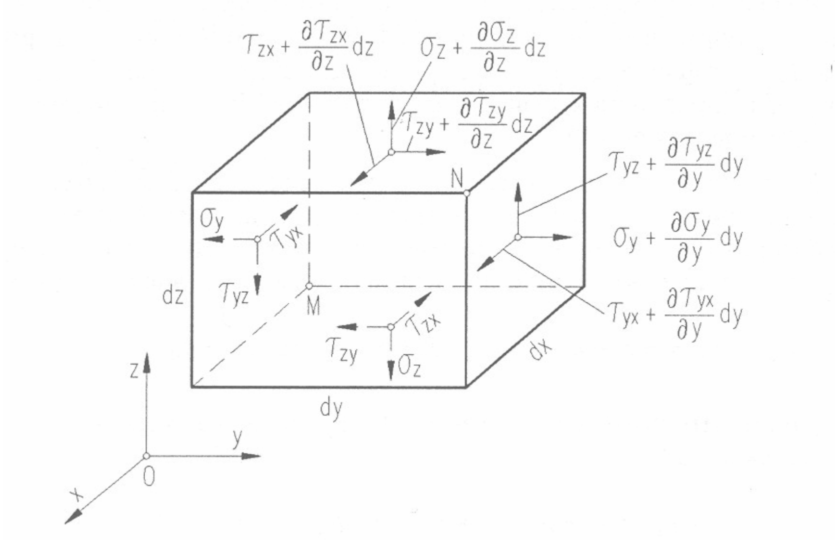


Fig. 6.3: Normal and tangential stresses

- *Tangential (shear) stresses:*

$$\tau_{xy} = \tau_{yx}\,; \qquad \tau_{xz} = \tau_{zx}\,; \qquad \tau_{yz} = \tau_{zy} \tag{6-4}$$

From the basic equation (6-1), the following relationships can be derived:

$$\tau_{xy} = \tau_{yx} = 2\mu\left[\frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] = \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \tag{6-5a}$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) \tag{6-5b}$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) \tag{6-5c}$$

- *Normal stresses:*

$$\sigma_x = -p + p_{tx}\,; \ \sigma_y = -p + p_{ty}\,; \qquad \sigma_z = -p + p_{tz} \tag{6-6}$$

p_{tx}, p_{ty}, p_{tz} - pressure increases due to the friction forces influence = *additional normal stresses*.

$p_{ti} = 0$ for ideal fluid $\Rightarrow$ $\sigma_x = -p$ etc (sign "-" for a direction towards the surface).

$\Rightarrow$ *Applying the approach for tangential stresses for* $p_{ti} = 0$ $\Rightarrow$

$$p_{tx} = \sigma_x + p = 2\mu\frac{\partial v_x}{\partial x} + \lambda\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) = 2\mu\frac{\partial v_x}{\partial x} + \lambda\,\mathrm{div}\,\vec{v} \tag{6-7a}$$

$\Rightarrow$ also

$$p_{ty} = \sigma_y + p = 2\mu \frac{\partial v_y}{\partial y} + \lambda \operatorname{div} \vec{v} \tag{6-7b}$$

$$p_{tz} = \sigma_z + p = 2\mu \frac{\partial v_z}{\partial z} + \lambda \operatorname{div} \vec{v} \tag{6-7c}$$

If the definition for *volumetric viscosity* is included:

$(\lambda + \frac{2}{3}\mu)$ - *volume viscousity* - see chapter 5.3.

Stokes assumed: $(\lambda + \frac{2}{3}\mu) = 0 \;\Rightarrow\; \lambda = -\frac{2}{3}\mu$

From the equations (6-7) $\Rightarrow$ *normal stresses*:

$$\sigma_x = -p + 2\mu \frac{\partial v_x}{\partial x} - \frac{2}{3}\mu \operatorname{div} \vec{v} \tag{6-8a}$$

$$\sigma_y = -p + 2\mu \frac{\partial v_y}{\partial y} - \frac{2}{3}\mu \operatorname{div} \vec{v} \tag{6-8b}$$

$$\sigma_z = -p + 2\mu \frac{\partial v_z}{\partial z} - \frac{2}{3}\mu \operatorname{div} \vec{v} \tag{6-8c}$$

- **Surface forces and friction forces**:

$\therefore$ *The surface forces can be easyly obtained from the previously defined stresses* $\Rightarrow$

Surface force per unit mass $dm = \rho dV$, in *"y"* direction (see *Fig. 6.3*):

$$S_y = \frac{1}{\rho}\left(\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \frac{\partial \tau_{xy}}{\partial x}\right) \tag{6-9a}$$

and in the *"x* and *"z"* directions:

$$S_x = \frac{1}{\rho}\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{zx}}{\partial z} + \frac{\partial \tau_{yx}}{\partial y}\right) \tag{6-9b}$$

$$S_z = \frac{1}{\rho}\left(\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{xz}}{\partial x}\right) \tag{6-9c}$$

$\Rightarrow$ By expressing the stresses from the equations (6-8), the *surface force in the "y"* dirrection will be:

$$S_y = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) + \frac{1}{3}\frac{\mu}{\rho}\operatorname{div} \vec{v} \tag{6-10}$$

With the vector notations:

$\Delta v_y = \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} = \nabla^2 v_y$, where, $\Delta = \nabla^2$ - *Laplacian operator*

$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$ - *vector operator* $\Rightarrow$ $\operatorname{grad} U = \nabla U$; $\operatorname{div} \vec{v} = (\nabla, \vec{v}) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

$$\Rightarrow \qquad S_y = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\nabla^2 v_y + \frac{1}{3}\frac{\mu}{\rho}\operatorname{div} \vec{v} = P_y + T_y \tag{6-11}$$

P_y - *normal surface force*; T_y - *tangential surface force (friction force)* - see chapter 1.4.

$$P_y = -\frac{1}{\rho}\frac{\partial p}{\partial y} \;; \qquad T_y = \frac{\mu}{\rho}\nabla^2 v_y + \frac{1}{3}\frac{\mu}{\rho}\operatorname{div} \vec{v} \tag{6-12}$$

$\therefore$ *Similar equation for "x" and "z" directions can be obtained* $\Rightarrow$ $S_y = P_y + T_y$; $S_z = P_z + T_z$

$\therefore$ *The entire normal surface force will be a vector sum:*

$$\vec{P} = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\vec{i} + \frac{\partial p}{\partial y}\vec{j} + \frac{\partial p}{\partial y}\vec{k}\right) = -\frac{1}{\rho}\nabla p \qquad (6\text{-}13)$$

$\Rightarrow$ The normal surface force has the same expression as for ideal (inviscid) fluid - see chapter 4.1.

$\therefore$ *The entire tangential friction surface force will be a vector sum:*

$$\vec{T} = \frac{\mu}{\rho}\nabla^2\vec{v} + \frac{1}{3}\frac{\mu}{\rho}\nabla\left(\nabla,\vec{v}\right) \qquad (6\text{-}14)$$

- **Navier-Stockes equations**:

❖ *Acting forces per unit mass* on a viscous fluid element with $dm = \rho dV$ (make comparison with chapter 4.1):

Inertial force per unit mass in N/kg $\Rightarrow$ $\vec{J} = \frac{d\vec{J}}{dm} = -\frac{d\vec{v}}{dt}$

Elementary resultant body force $\vec{R}$ in N/kg $\Rightarrow$ $\vec{R} = \frac{\partial U}{\partial x}\vec{i} + \frac{\partial U}{\partial y}\vec{j} + \frac{\partial U}{\partial z}\vec{k} = \mathrm{grad}\,U = \nabla U$

Tangential surface force in N/kg $\Rightarrow$ $\vec{T} = \frac{\mu}{\rho}\nabla^2\vec{v} + \frac{1}{3}\frac{\mu}{\rho}\nabla\left(\nabla,\vec{v}\right) = \frac{\mu}{\rho}\Delta\vec{v} + \frac{1}{3}\mathrm{grad}\left(\mathrm{div}\,\vec{v}\right)$

Normal surface force in N/kg $\Rightarrow$ $\vec{P} = -\frac{1}{\rho}\nabla p = -\frac{1}{\rho}\mathrm{grad}\,p$

According the *D'Alembert's principle* for dynamic equilibrium $\Rightarrow$:

$$\vec{J} + \vec{R} + \vec{P} + \vec{T} = 0$$

$\Rightarrow$ *Navier-Stockes equations* in vector notations:

$$\frac{d\vec{v}}{dt} = \nabla U - \frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2\vec{v} + \frac{1}{3}\frac{\mu}{\rho}\nabla\left(\nabla,\vec{v}\right) \qquad (6\text{-}15)$$

i.e. $$\frac{d\vec{v}}{dt} = \vec{R} - \frac{1}{\rho}\mathrm{grad}\,p + \frac{\mu}{\rho}\Delta\vec{v} + \frac{1}{3}\frac{\mu}{\rho}\mathrm{grad}\left(\mathrm{div}\,\vec{v}\right) \qquad (6\text{-}15a)$$

$\frac{\mu}{\rho} = \nu$ - *kinematic viscosity.*

$\therefore$ *Governing equations of viscous fluid laminar flow (mathematical model)*
= *Navier-Stockes equations* (6-15) + *continuity equation* (6-2) + energy equation (6-3).

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\vec{v}) = 0;\ \text{i.e.}\ \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = 0 \qquad (6\text{-}2)$$

$$\rho = f(p, T, c_p, k, ...,);\quad \text{for barotropic fluid}\quad \rho = f(p) \qquad (6\text{-}3)$$

⇒ *The Navier-Stockes vector equation can be expressed as three scalar Navier-Stockes partial differential equations - equations of motion in Cartesian coordinates (Oxyz):*

$$\rho \frac{dv_x}{dt} = \rho \frac{\partial U}{\partial x} - \frac{\partial p}{\partial x} + \frac{\mu}{3} \frac{\partial \theta}{\partial x} + \mu \nabla^2 v_x \tag{6-16a}$$

$$\rho \frac{dv_y}{dt} = \rho \frac{\partial U}{\partial y} - \frac{\partial p}{\partial y} + \frac{\mu}{3} \frac{\partial \theta}{\partial y} + \mu \nabla^2 v_y \tag{6-16b}$$

$$\rho \frac{dv_z}{dt} = \rho \frac{\partial U}{\partial z} - \frac{\partial p}{\partial z} + \frac{\mu}{3} \frac{\partial \theta}{\partial y} + \mu \nabla^2 v_z \tag{6-16c}$$

where is:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}; \qquad \theta = \operatorname{div} \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}; \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

In case of *2-D incompressible fluid laminar flow* ⇒ $\theta = \operatorname{div} \vec{v} = 0$; $v_z = 0$; $\frac{\partial}{\partial z} = 0$

⇒ *the governing equations*:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{6-17a}$$

$$\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \frac{\partial U}{\partial x} - \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\mu}{\rho} \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right) \tag{6-17b}$$

$$\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = \frac{\partial U}{\partial y} - \frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\mu}{\rho} \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} \right) \tag{6-17c}$$

∴ *Analytical solution of the system of governing partial differential equations is possible only for a few cases of laminar, steady flow of incompressible fluid.*

Several approximations are introduced in these cases. ⇒ *Results differ from reality.*

6.3. Fundamental concepts and solutions of the governing equations for some cases of laminar flow

- **Steady laminar flow between parallel plates -** *see Fig. 6. 4*:

Approximations:

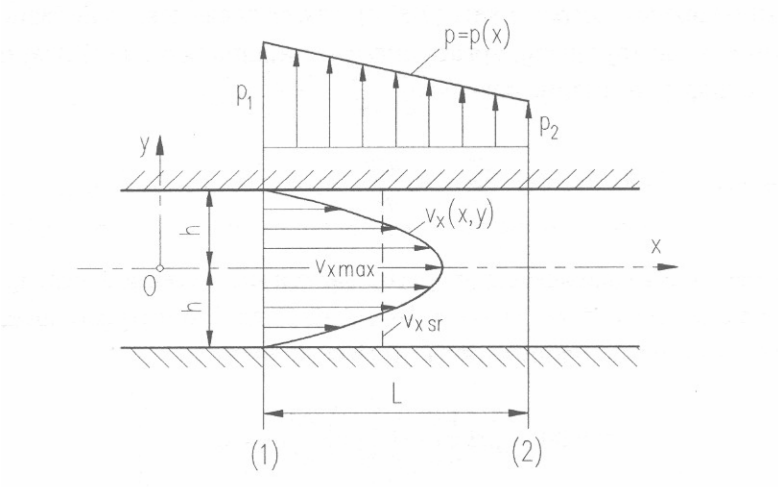
steady laminar 2-D flow, incompressible viscous fluid, body forces are neglected ⇒

$\frac{\partial}{\partial t} = 0$; $v_y = 0$; $v_x = v_x(x, y)$; $U = 0$. ⇒ the system (6-17) is simplified ⇒

$$\frac{\partial v_x}{\partial x} = 0 \tag{6-18a}$$

$$v_x \frac{\partial v_x}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\mu}{\rho} \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right) \tag{6-18b}$$

$$0 = -\frac{\partial p}{\partial y} \tag{6-18c}$$



$v_{xsr} = v_{ave}$; $v_{x\max} = v_{\max}$

Fig.6.4: Flow between parallel plates

$\Rightarrow$ The system of the partial differential equations (6-18) can be transformed to one ordinary differential equation:

$$\frac{dp}{dx} = \mu \frac{d^2 v}{dy^2} \tag{6-19}$$

Because, $p = p(x)$, $v_x = v(y) = v$, after the integration of (6-19), following the boundary conditions from *Fig. 6.4* $\Rightarrow$ *velocity change/velocity distribution* along "*y*":

$$v(y) = -\frac{1}{2\mu}\left(h^2 - y^2\right)\frac{dp}{dx} \tag{6-20}$$

Maximum velocity (at $y = 0$): $$v_{\max} = -\frac{h^2}{2\mu}\frac{dp}{dx} \tag{6-21}$$

Average velocity: $$v_{ave} = \frac{2}{3} v_{\max} = -\frac{h^2}{3\mu}\frac{dp}{dx} = \frac{Q}{A} \tag{6-22}$$

Where is: $A = 2bh$ and $Q = b\int_{-h}^{+h} v(y)dy = -\frac{2}{3\mu} bh^3 \frac{dp}{dx}$

The pressure distribution is assumed to be a straight line (see *Fig. 6.4*) $\Rightarrow$ $\frac{dp}{dx} = const$ $\Rightarrow$

$$p_1 - p_2 = \frac{3\mu L}{h^2} v_{ave} \tag{6-23}$$

- **Steady laminar flow in a circular tube of a constant diameter** - *see Fig. 6. 7*:

Same approximations (steady laminar 2-D flow, incompressible viscous fluid) and similar approach as in the previous case:

$\therefore$ After simplification and transformation of the system governing partial differential equations in cylindrical coordinates, an ordinary differential equation can be obtained as well.

The integration with boundary conditions as on *Fig. 6.7* gives *the velocity distribution* :

$$v(r) = \frac{k}{4\mu}\left[\left(\frac{D}{2}\right)^2 - r^2\right] = -\frac{1}{4\mu}\frac{dp}{dz}\left[\left(\frac{D}{2}\right)^2 - r^2\right] \tag{6-24}$$

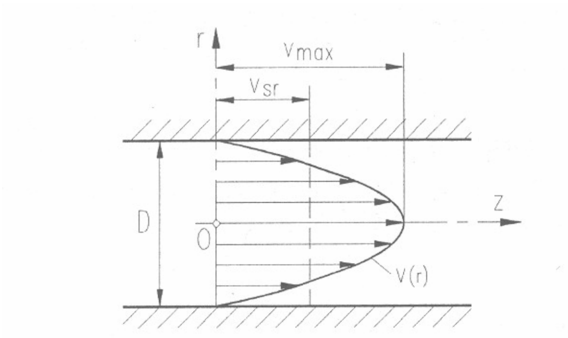
Maximum velocity (at $r = 0$): $$v_{\max} = -\frac{D^2}{16\mu}\frac{dp}{dz} \qquad (6\text{-}25)$$

Average velocity: $$v_{ave} = \frac{1}{2}v_{\max} = \frac{D^2}{32\mu}\left(-\frac{dp}{dz}\right) = \frac{Q}{A} \qquad (6\text{-}26)$$

Where is: $A = \pi D^2/4$; $Q = \int_{r=0}^{D/2} v(r) 2r\pi dr$

The pressure distribution along the pipe: $\frac{dp}{dz} = -k = const$ $\Rightarrow$ along a length L of the pipe $\Rightarrow$

$$\Delta p = p_1 - p_2 = \frac{32\mu L v_{ave}}{D^2} \qquad (6\text{-}27)$$



$v_{sr} = v_{ave}$

Fig. 6-7: Steady laminar flow in a circular tube of a constant diameter

6.4. Fundamental concepts and equations for creeping motion and two-dimensional boundary layer

- **Creeping flow**

Creeping fluid flow or Stokes flow (named after George Gabriel Stokes) is a type of incompressible fluid flow where inertial forces are small compared with viscous forces. The Reynolds number is low, i.e. $\mathrm{Re} \to 0$.

This is a typical situation in flows where the fluid velocities are very slow, the viscosities are very large, or the length-scales of the flow are very small, such as in *Microelectromechanical systems* (MEMS) devices or in the flow of *viscous polymers*.

$\therefore$ *Inertial forces and body are neglected* $\Rightarrow$ *Stokes equations* :

$$\mathrm{grad}\, p = \mu \Delta \vec{v} \qquad (6\text{-}28)$$

Using the continuity equation for incompressible fluid flow, $\mathrm{div}\,\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \Rightarrow$

Laplace equation for pressure:

$$\Delta p = \nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = 0 \qquad (6\text{-}29)$$

$\Rightarrow$ the pressure can be obtained directly from the Laplace equation (6-29), taking the boundary conditions into account.

- *Creeping flow around a sphere* - see *Fig. 6.9*:

This is a characteristic example!

The Stockes equation (6-28) is solved in spherical coordinate system ($r,\ \theta, \varphi$) - see *Fig. 6.9*, using the boundary conditions:

for $r = R \;\Rightarrow\; v = 0$; for $r \to \infty \;\Rightarrow\; v_x = v_\infty$ and $p_x = p_\infty$

∴ *Stokes derived the acting force of the fluid on the sphere in the "x" direction - F_x = called the Drag force (F_D)!* :

$$F_x = 6\pi\mu R v_\infty \tag{6-30a}$$

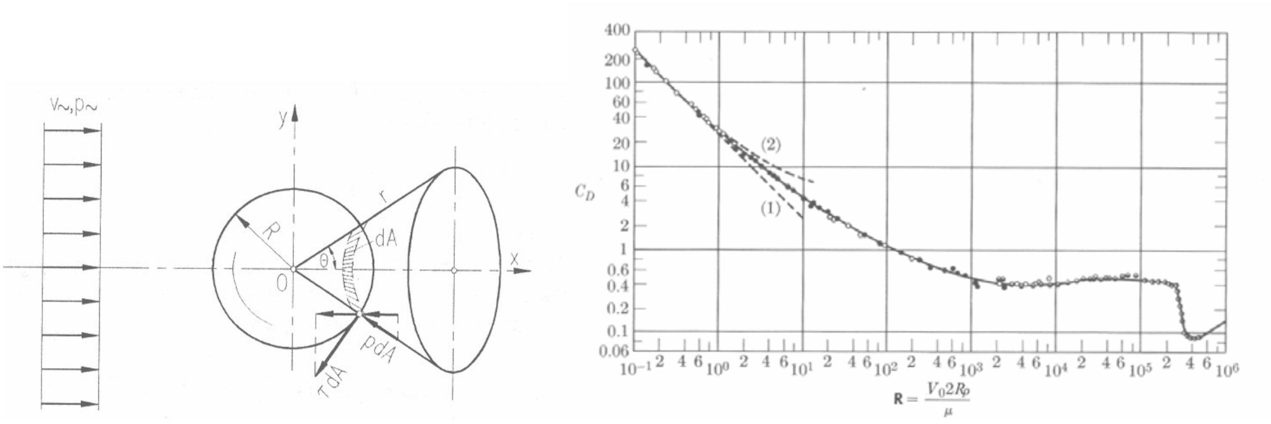
or

$$F_x = F_D = C_D \rho \frac{v_\infty^2}{2} A \tag{6-30b}$$

Where is: $A = \pi R^2$; $C_D = \dfrac{24}{\text{Re}}$ - *theoretical Ctokes Drag coefficient*; $\text{Re} = \dfrac{v_\infty 2R\rho}{\mu}$

∴ *However, since the inertial forces are neglected* $\Rightarrow$ $C_D \neq \dfrac{24}{\text{Re}}$.

∴ C_D has to be obtained with experiment - see *Fig. 6.9b*.



a) *b)* $C_D = f(\text{Re})$ - solid line = experiments; (1) - Stokes' equation; (2) - Oseen's equation

Fig. 6.9: Creeping flow around a sphere

- **Two-dimensional boundary layer**

A boundary layer is that layer of fluid in the immediate vicinity of a bounding surface- Fig. 6.10a,b.

In the Earth's atmosphere, *the planetary boundary layer* is the air layer near the ground affected by diurnal heat, moisture or momentum transfer to or from the surface.

On an aircraft wing the boundary layer is the part of the flow close to the wing.

The *boundary layer effect* occurs at the field region in which all changes occur in the flow pattern. The boundary layer distorts surrounding nonviscous flow.
It is a phenomenon of viscous forces. This effect is related to the Reynolds number.

∴ Some conclusions:

- *With real fluids there is no "slip" at the riigid boundaries. The fluid velocity relative to the boundary is zero (see Fig. 6.10).*
- *The velocity gradient and shear stress have maximum values at the boundaries.*
- *Significant viscous shear occurs only within a thin layer next to the boundary (called "boundary layer" . Outside this layer viscous shear becomes small.*
- *Inside the layer, the viscous effects override the inertia effects.*
- *The stream lines of the main flow beyond the boundary layer conform essentially to a potential flow!!!*

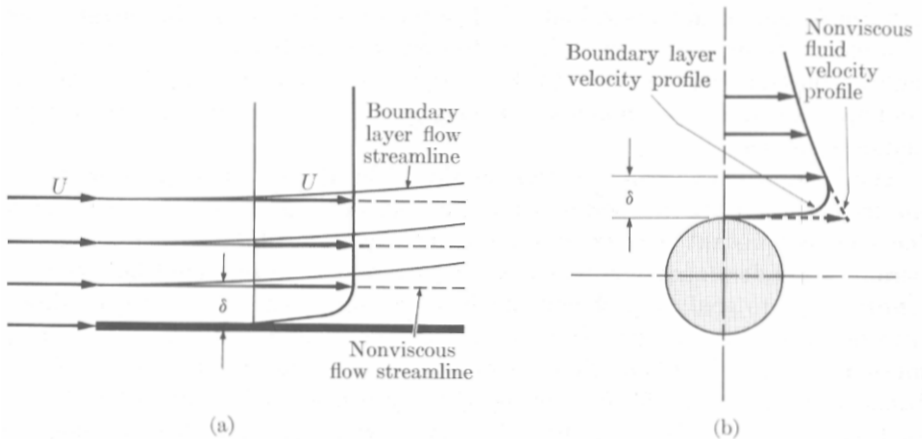


Fig. 6.10 a,b: Boundary layer versus slip flow: (a) flat plate ;(b) cylinder

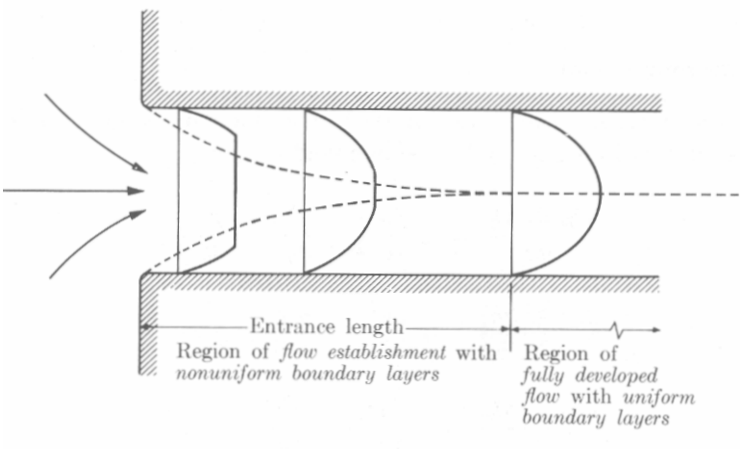


Fig. 6.10 c: Boundary layers in ducts

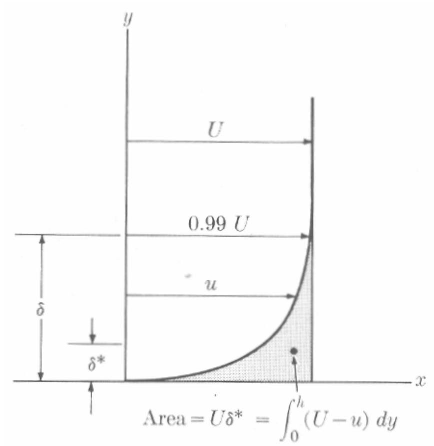


Fig. 6.10 d:Boundary layer thickness

Numerous theoretical and experimental investigations are realized concerning the boundary layer phenomena!
General classification: laminar and turbulent boundary layer (see Fig. 6.11).

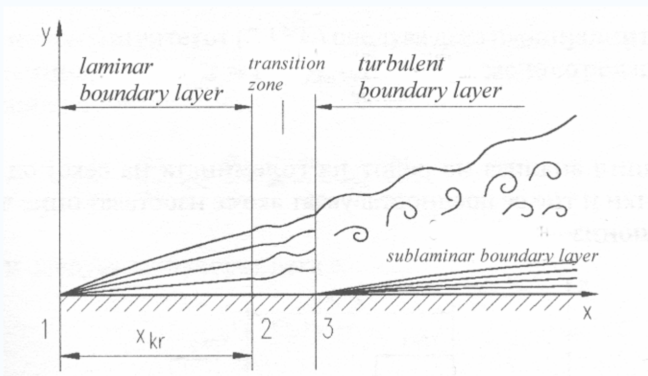


Fig. 6.11: Laminar and turbulent boundary layer

In this chapter only a basic approach to the laminar boundary layer is presented.

Laminar boundary layers come in various forms and can be loosely classified according to their structure and the circumstances under which they are created.

$\therefore$ *For 2-D laminar boundary layer of creeping incompressible fluid flow (inertial and body forces are neglected)* $\Rightarrow$

Prandtl derived boundary layer equations from the governing equations for 2-D incompressible fluid laminar flow (equations (6-17)):

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{6-31a}$$

$$\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right) \tag{6-31b}$$

$\frac{\partial v_x}{\partial t} = 0$ for steady flow.

$\Rightarrow$ The so-called *displacement thickness* of an imaginary boundary layer δ^* can be obtained, considering the continuity of the mass flow rate adjacent to the boundary layer (see *Fig. 6.11*):

$$\rho U \delta^* = \rho \int_0^h (U - u) dy$$

where is: $u = v_x$; U - free stream velocity; $\rho U \delta^*$ - mass flow rate in the absence of boundary layer.

$$\delta^* = \int_0^h \left(1 - \frac{u}{U}\right) dy \tag{6-32}$$

$\therefore$ Difficulties in numerical obtaining of the *overall boundary layer thickness* $\delta \Rightarrow$ experiments , approximations:

$\therefore$ δ *is defined as the distance to the point where* $v_x = 0.99U$ *(see Fig. 6.11).*

6.5. The notion of resistance, drag, and lift

The investigation of the drag and lift concepts are very important for various fields oh Fluid Mechanics application: aeronautics, turbo machinery, multicomponents flows, chemical reactions etc.

Drag (sometimes called resistance) is the force that resists the movement of a solid object through a fluid in the direction of its movement - in this case the object is moving in a quiescent fluid.

Drag force (F_D) *can be also defined as the acting force of the fluid flow on a immersed body, in the direction of the flow relative velocity* V_0 - see *Fig. 6.12*.

The total drag force F_D can be expressed with its *components* as (see *Fig.6.12*):

$$F_D = F_{Df} + F_{Dp} \tag{6-33}$$

frictional drag: $$F_{Df} = \int_S \tau_0 \sin\varphi dS \tag{6-34}$$

pressure drag: $$F_{Dp} = -\int_S p \sin\varphi dS \tag{6-35}$$

S - total surface area.

However, using the *Stokes approaches* (see equation (6-30b)), these components can be expressed as:

$$F_{Df} = C_{Df}\rho\frac{V_0^2}{2}A_f \tag{6-36}$$

$$F_{Dp} = C_{Df}\rho\frac{V_0^2}{2}A_p \tag{6-37}$$

C_{Df} and C_{Dp} - *corresponding drag (resistance) coefficients;* A_f and A_p - *reference areas*.
Usually, A_f - the actual area over which the shear stresses act, e.g. the planform area of a wing or hydrofoil (see Fig. 6.13);
A_p - the frontal area of a wing or hydrofoil (see Fig. 6.13).

∴ *The total drag force can also be defined as:*

$$F_D = C_D\rho\frac{V_0^2}{2}A \tag{6-38}$$

$$C_D = C_{Df} + C_{Dp} \tag{6-39}$$

A - frontal area normal to $V_0 \Rightarrow A = A_p$

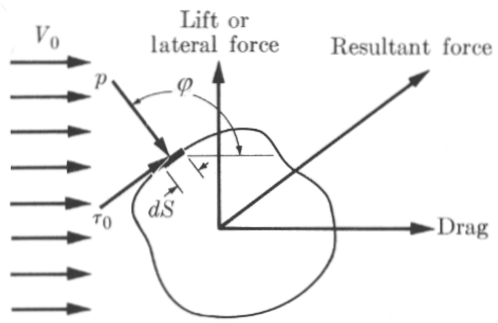


Fig. 6.12: Definition diagram for flow-induced forces

Lift force is the sum of all the fluid dynamic forces on a body perpendicular to the direction of the external flow approaching that body - see Fig. 6.12 and Fig. 6.13.

The lift force, lifting force or simply *lift* can also be defined as a *mechanical force generated by solid objects as they move through a fluid*.

While many types of objects can generate lift, the most common and familiar object in this category is the airfoil, a relatively flat object of which the common airplane wing is an example - *Fig. 6.13*.

For the lift force it is not customary to separate the frictional and pressure components. For bodies like the hydrofoil (*Fig. 6.13*), designed particularly for useful lift, the *lift force* is primarily a pressure-component effect.

∴ *The total lift is defined as:*

$$F_L = L = C_L\rho\frac{V_0^2}{2}A \tag{6-40}$$

C_L - lift coefficient;
A - the planform area of a wing (largest projected area of the body, or the projected area normal to V_0.

∴ C_D and C_L are usually experimentally obtained. Theoretical approaches exist but with many approximations.

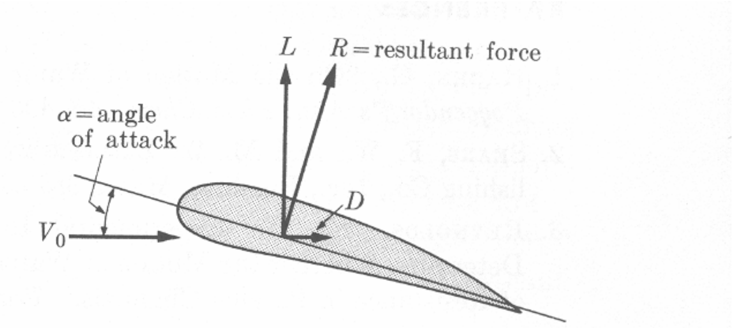


Fig. 6.13:Lift and drag on a hydrofoil section

6.6. Basic concepts of incompressible viscous fluid turbulent flow

Turbulent flow is a flow regime characterized by chaotic, stochastic property changes.

Turbulent flow - a flow field that cannot be described with streamlines in the absolute sense. However, time-averaged streamlines can be defined to describe the average behavior of the flow. In turbulent flow, the inertia stresses dominate over the viscous stresses, leading to small-scale chaotic behavior in the fluid motion.

∴ *Turbulent flows are more common in the nature and more significant*

- **Reynolds experiment and Reynolds number**

During the later half of the 19th century, Osborne Reynolds demonstrated the difference in laminar and turbulent flow and developed an equation to predict the transition from one flow regime to the other.

Experiment includes (*Fig. 6.16*):
- Water supply tank with clear test section tube and "bell mouth" entrance.
- Dye injector with needle valve control for precision metering of dye.
- Rotometer flow meter to measure water flow rate.
- One bottle of dye.

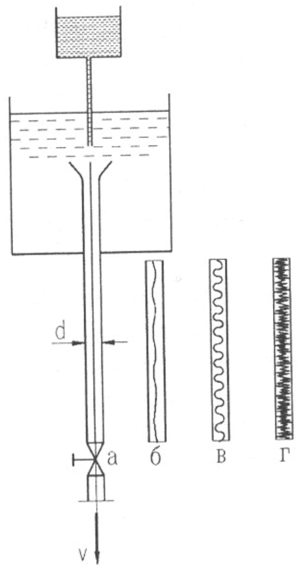


Fig. 6.16: Scheme of the Reynolds experiment

As shown on *Fig. 6.16*, Reynolds injected a fine stream of dye into water flowing from a large tank into a glass tube. ⇒:

- With *low flow rates* through the tube (small velocities), the dye stream persisted as a straight streak ⇒ ∴ the water moved in parallel stream lines or laminas ⇒ *laminar flow*.
- As the flow was increased above a certain critical rate, the dye streak broke into irregular vortices and then (for very high velocities) mixed laterally throughout the cross section ⇒ *turbulent flow*.
- See also *Fig. 6.2*.

The dimensionless *Reynolds number* - Re is an important parameter in the equations that describe whether flow conditions lead to laminar or turbulent flow:

$$\frac{\text{inertia force/mass}}{\text{frictional force/mass}} \propto \text{Reynolds number} \qquad (6\text{-}41)$$

$\mathrm{Re} < \mathrm{Re}_{cr}$ - *laminar flow*; $\mathrm{Re} > \mathrm{Re}_{cr}$ - *turbulent flow*.

Re_{cr} - *critical Reynolds number*.

Re is one of the most *important dimensionless numbers* in fluid dynamics and is used, usually along with other dimensionless numbers, to provide a criterion for determining *dynamic similitude*.

Concerning the definition (6-41) in *the theory of similarity* (see chapter 7) the following expression is derived:

$$\mathrm{Re} = \frac{\rho v_0 l}{\mu} = \frac{v_0 l}{\nu} \qquad (6\text{-}42)$$

v_0 - characteristic velocity; l - characteristic length.

For flow in circular pipes ⇒

$$\mathrm{Re} = \frac{\rho v_m d}{\mu} \qquad (6\text{-}43)$$

$v_m = Q/A$ - mean velocity; d - pipe diameter.

The transition between laminar and turbulent flow is often indicated by a *critical Reynolds number* (Re_{cr}), which depends on the exact flow configuration and must be determined experimentally.

$\mathrm{Re} < \mathrm{Re}_{cr}$ - *laminar flow*; $\mathrm{Re} > \mathrm{Re}_{cr}$ - *turbulent flow*.

For example:

$\mathrm{Re}_{cr} = 2320$ - *critical Reynolds number for flow in pipes*.

$\mathrm{Re}_{cr} = 1160$ - *critical Reynolds number for flow wide channnels (the depth is characteristic length).*

However, within a certain range around the critical Re value, there is a region of gradual transition where the flow is neither fully laminar nor fully turbulent, and predictions of fluid behavior can be difficult.
⇒ Many engineers will avoid any pipe configuration that falls within the range of Reynolds numbers from about 2000 to 3000 to ensure that the flow is either laminar or turbulent.

- **Velocity in turbulent flow**

Turbulent flow has a random nature, making it difficult to describe exactly.
It can be describe by a set of statistical properties.
For this purpose, it is convenient to define the term of *instantaneous flow properties*

$$u = \overline{u} + u' \; ; \quad v = \overline{v} + v' \; ; \quad w = \overline{w} + w' \; ; \quad p = \overline{p} + p' \quad \text{etc.} \tag{6-44}$$

e.g.: v - *instantaneous velocity*; $\overline{v}$ - *mean value*; v' - *fluctuating component*; ⇒ see *Fig. 6.17*.

Here: $u = v_x$; $v = v_y$; $w = v_z$ - instantaneous velocities in the corresponding directions *x,y,z*.

$$\overline{v} = \frac{1}{T}\int_0^T v\,dt \tag{6-45}$$

$$\Rightarrow \quad \overline{v'} = \frac{1}{T}\int_0^T v'\,dt \; ; \qquad \overline{u'v'} = \frac{1}{T}\int_0^T u'v'\,dt \; ; \quad \text{etc.} \tag{6-46}$$

⇒ *Kinetic energy of turbulence per unit mass*:

$$\frac{\text{average KE of turbulence}}{\text{mass}} = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right) \tag{6-47}$$

⇒ ∴ *For other properties the same approach can be applied!*

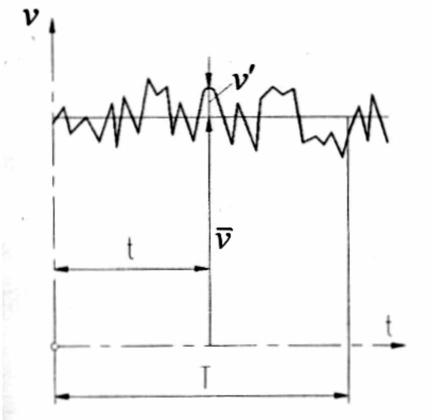


Fig. 6.17: Turbulent flow instantaneous velocity

Introducing the mean values, as given with equations (6-45) to (6-47), it is possible to obtain partial differential equations for mean flow of incompressible viscous fluid flow (from the governing equations derived in chapter 6.2).

Reynolds converted the equations of motion for incompressible viscous fluid flow into such form ⇒ *Reynolds equations for incompressible turbulent flow*.
However, the introduced approximations make that the theoretically predicted behavior is different from the real behavior - the true details of the fluctuations are not established.

- **Governing equations for turbulent flow**

The governing equations derived in chapter 6.2, *Navier-Stockes equations* (6-16) + *continuity equation* (6-2) + *energy equation* (6-3) , are general and valid for turbulent flow as well.

∴ *The mathematical model of a turbulent compressible fluid flow (general case) can be expressed with the following system of partial differential equations:*

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0 \qquad (6\text{-}48)$$

$$\rho\frac{Du}{Dt}=\rho X-\frac{\partial p}{\partial x}+(\lambda+\mu)\frac{\partial \theta}{\partial x}+\mu\nabla^2 u \qquad (6\text{-}49)$$

$$\rho\frac{Dv}{Dt}=\rho Y-\frac{\partial p}{\partial y}+(\lambda+\mu)\frac{\partial \theta}{\partial y}+\mu\nabla^2 v \qquad (6\text{-}50)$$

$$\rho\frac{Dw}{Dt}=\rho Z-\frac{\partial p}{\partial z}+(\lambda+\mu)\frac{\partial \theta}{\partial z}+\mu\nabla^2 w \qquad (6\text{-}51)$$

$$\rho c_p\frac{DT}{Dt}-\frac{Dp}{Dt}=\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\Phi \qquad (6\text{-}52)$$

ρ - density;
$u=v_x$, $v=v_y$ and $w=v_z$ - velocities in x, y and z directions;
For turbulent flow: $u=\overline{u}+u'$; $v=\overline{v}+v'$; $w=\overline{w}+w'$; $p=\overline{p}+p'$ etc.
t - time;
p - pressure;
T - temperature;
k - heat conductivity coefficient;
c_p - specific heat at constant pressure;
X, Y, Z - body force components per unit mass in x, y and z direction;
$\Phi=(\lambda+\frac{2}{3}\mu)\theta^2$ - dissipation function;
$(\lambda+\frac{2}{3}\mu)$ - *volumetric viscosity;* μ - dynamic viscosity;
$\theta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$ - local rate of volumetric dilatation;
$\frac{D}{Dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}$ - operator for differentiation;
$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ - Laplace's operator.

∴ *It is obvious that for turbulent flow the governing equations are scientifically complex.*

∴ *Analytical solution of the system of governing partial differential equations is possible only for a few cases of laminar, steady flow of incompressible fluid* (see chapter 6.3).

∴ *Several approximations are introduced in these cases* ($\theta=0$; $\rho=\cos nt$; $T=const$; etc.).
⇒ *Results differ from reality.*

∴ *Exact analytical solution of the mathematical model defined with the system partial differential equations (6-48) to (6-52) is not possible.*

6.7. Concepts for solving governing equations of viscous fluid flow

For engineering problems solving, two general methods are available.

- *theoretical, and*
- *experimental.*

In the engineering analysis, and especially research work, for every problem it is necessary to figure out the use of one or the other method

For most of the engineering problems the implementation of both methods is necessary.

Which one will be used more (or less) depends of the nature of the problem and the available knowledge.

The theory and the experiment have to be compatible. ⇒ *more efficiency in solving the problems*.

- **Features of theoretical methods**

The result of the theoretical method is definition of corresponding *mathematical model*, which gives the description of the investigated problem.

If analytical solutions of the mathematical model are possible ⇒ overall results are obtained, which will be valid for different conditions.

∴ *The effort has to be in finding analytical solutions first of all.*

However, analytical solution of the system of governing partial differential equations is possible only for a few cases (see previous conclusions).

⇒ Defining of corresponding *numerical model.*
⇒ Several *approximations are* introduced in the process of numerical model definition.
⇒
∴ The numerical solutions give predictions for the corresponding process behavior.

The use of sophisticated PC (even so-called "super computers") and software packages enable solving of such numerical models, for which extremely long execution time was needed in the past (or it was impossible to be solved).

The features of the theoretical method can be summarized as follows:

1. Often give results that are of general use rather than for restricted application.
2. Invariably require the application of simplifying assumptions. Thus not the actual physical system but rather a simplified "mathematical model" of the system is studied. This means the theoretically predicted behavior is always different from the real behavior.
3. In some cases, may lead to complicated mathematical problems. This has blocked theoretical treatment of many problems in the past. Today, increasing availability of high-speed computing machines allows theoretical treatment of many problems that could not be so treated in the past.

4. Besides theoretical knowledge; require only pencil, paper, computing machines, etc. Extensive laboratory facilities are not required. (Some computers are very complex and expensive, but they can be used for solving all kinds of problems. Much laboratory equipment, on the other hand, is special-purpose and suited only to a limited variety of tasks.)
5. No time delay engendered in building models, assembling and checking instrumentation, and gathering data.

Concerning theoretical method for solving *governing equations of viscous fluid flow*:

- *Turbulence modeling* is the area of physical modeling where a simpler mathematical model than the full time dependent Navier-Stokes equations is used to predict the effects of turbulence.

- *Reynolds-averaged Navier-Stokes equations (RANS)* is the oldest approach to turbulence modeling. An ensemble version of the governing equations is solved, which introduces new *apparent stresses* known as Reynolds stresses.

⇒ Mathematically, turbulent flow is represented via *Reynolds decomposition*, in which the flow is broken down into the sum of a steady component and a perturbation component
⇒ see previously defined *instantaneous, mean and fluctuating flow properties* - equations (6-44) to (6-47).

⇒ Derivation of the *Reynolds-averaged Navier-Stokes (RANS) equations*, which are time-averaged equations of motion for fluid flow. They have been primarily used while dealing with turbulent flows.

- Joseph Boussinesq was the first practitioner of this, introducing the concept of eddy viscosity. In this model, the additional turbulent stresses are given by augmenting the molecular viscosity with an eddy viscosity. This can be a simple constant eddy viscosity (which works well for some free shear flows such as axisymmetric jets, 2-D jets, and mixing layers).
- Later, Ludwig Prandtl introduced the additional concept of the mixing length, along with the idea of a boundary layer. For wall-bounded turbulent flows, the eddy viscosity must vary with distance from the wall, hence the addition of the concept of a 'mixing length'. In the simplest wall-bounded flow model, the eddy viscosity is given by the equation.

- However, since it is believed that turbulent flows obey the Navier-Stokes equations. *Direct Numerical Simulation* (DNS), based on the incompressible Navier-Stokes equations, makes it possible to simulate turbulent flows with moderate Reynolds numbers (restrictions depend on the power of computer and efficiency of solution algorithm). The results of DNS agree with the experimental data. The DNS is widly applied in *Computational fluid dynamics approach (CFD).*

- **Experimental and semi-empirical approach**

Especially, for problems being on the edge of knowledge, i.e. there are now enough adequate theoretical descriptions and predictions, extensive experimental investigations are needed.

However, the links between the existing theory and the experiment have to be defined ⇒ *Dimensional analysis* and *Theory of similarity* are of great help.

For conducting the experimental method, defining and realization of a *physical model* is needed

The physical model has to be similar to the original (prototype) as much as possible.

A corresponding laboratory installation has to be constructed for the defined physical model ⇒ experiments and measurements of the governing properties would be performed.

The features of the experimental method can be summarized as follows:

1. Often give results that apply only to the specific system being tested. However. techniques such as dimensional analysis may allow some generalization.
2. No simplifying assumptions necessary if tests are run on an actual system. The true behavior of the system is revealed.
3. *Accurate* measurements necessary to give a true picture. This may require expensive and complicated equipment. *The characteristics of all the measuring and recording equipment must be thoroughly understood.*
4. Actual system or a scale model required. If a scale model is used, similarity of all significant features must be preserved.
5. Considerable time required for design, construction, and debugging of apparatus.

∴ The experimental method can help in resolving the problems of introducing the approximations in the process of solving the mathematical/numerical model.

⇒

Types of problems that can be resoled by use of the experimental model;

1. Testing the validity of theoretical predictions based on simplifying assumptions; improvement of theory, based on measured behaviour.
   ⇒ *semi-empirical approach for solving the governing equations*.
2. Formulation of generalized *empirical relationships* in situations where no adequate theory exists.
   Example: determination of friction factor for turbulent pipe flow.
3. Determination of material, component, and system parameters, variables, and performance indices.
   Examples: determination of yield point of certain alloy steel, speed-torque curves for an electric motor, thermal efficiency of a steam turbine.
4. Study of phenomena with hopes of developing a theory.
   Example: electron microscopy of metal fatigue cracks.
5. Solution of mathematical equations by means of analogies.
   Example: solution of shaft torsion problems by measurements on soap bubbles.

- **CFD approach**

The development of the Numerical analysis, and especially the development and application of the sophisticated computers and software, have introduced numerical methods for solving the governing equations. These methods can be classified in general as:

- integral method,
- method of finite elements/differences,
- method of finite volumes.

A *direct numerical simulation (DNS)* is a simulation in *computational fluid dynamics* (*CFD*) in which the Navier-Stokes equations are numerically solved without any turbulence model. This means that the whole range of spatial and temporal scales of the turbulence must be resolved. All the spatial scales of the turbulence must be resolved in the computational mesh, *Direct numerical simulation (DNS)* captures all of the relevant scales of turbulent motion, so no model is needed for the smallest scales. This approach is extremely expensive, if not intractable, for complex problems on modern computing machines, hence the need for models to represent the smallest scales of fluid motion.

∴

The behavior of fluid flow is described by well-established partial differential equations – *governing equations*.

Except for very simple conditions, these equations need to be solved numerically with the aid of computers.

To this end, the predefined flow domain is covered by a *numerical mesh*, which defines *nodes* at mesh cross-sections and *finite volumes* or *finite elements* which are patches of area of volume cells around nodes or between consecutive mesh lines.

The differential flow-governing equations are then approximated, using numerical discretisation schemes, as sets of algebraic equations, each pertaining to a node, finite volume or finite element. The collection of coupled algebraic equation are then solved, by linear-algebra methods, on a computer to yield discrete values of velocity and pressure at mesh nodes.

The collection of theoretical, numerical and computational techniques that facilitate this process is called *Computational Fluid Dynamics*.

Computational fluid dynamics (CFD) is one of the branches of fluid mechanics that uses numerical methods and algorithms to solve and analyze problems that involve fluid flows. Computers are used to perform the millions of calculations required to simulate the interaction of fluids with the complex surfaces used in engineering.

What use is CFD?

Knowing how fluids will flow, and what will be their quantitative effects on the solids with which they are in contact, *CFD* assists in:

- building-services engineers and architects to provide comfortable and safe human environments;
- power-plant designers to attain maximum efficiency, and reduce release of pollutants;
- chemical engineers to maximize the yields from their reactors and processing equipment;
- land-, air- and marine-vehicle designers to achieve maximum performance, at least cost;
- risk-and-hazard analysts, and safety engineers, to predict how much damage to structures, equipment, human beings, animals and vegetation will be caused by fires, explosions and blast waves.

CFD-based flow simulations enable:

- metropolitan authorities need to determine where pollutant-emitting industrial plant may be safely located, and under what conditions motor-vehicle access must be restricted so as to preserve air quality;
- meteorologists and oceanographers to foretell winds and water currents; - hydrologists and others concerned with ground-water to forecast the effects of changes to ground-surface cover, of the creation of dams and aquaducts on the quantity and quality of water supplies;
- petroleum engineers to design optimum oil-recovery strategies, and the equipment for putting them into practice;
- ... and so on.

Within a few years, it is to be expected, surgeons will conduct operations which may affect the flow of fluids within the human body (blood, urine, air, the fluid within the brain) only after their probable effects have been predicted by CFD methods.

However, even with simplified equations and high-speed supercomputers, still approximate solutions can be achieved in many cases.
More accurate software that can accurately and quickly simulate even complex scenarios such as transonic or turbulent flows are an ongoing area of research. Validation of such software is often performed using experiments on a physical model. .

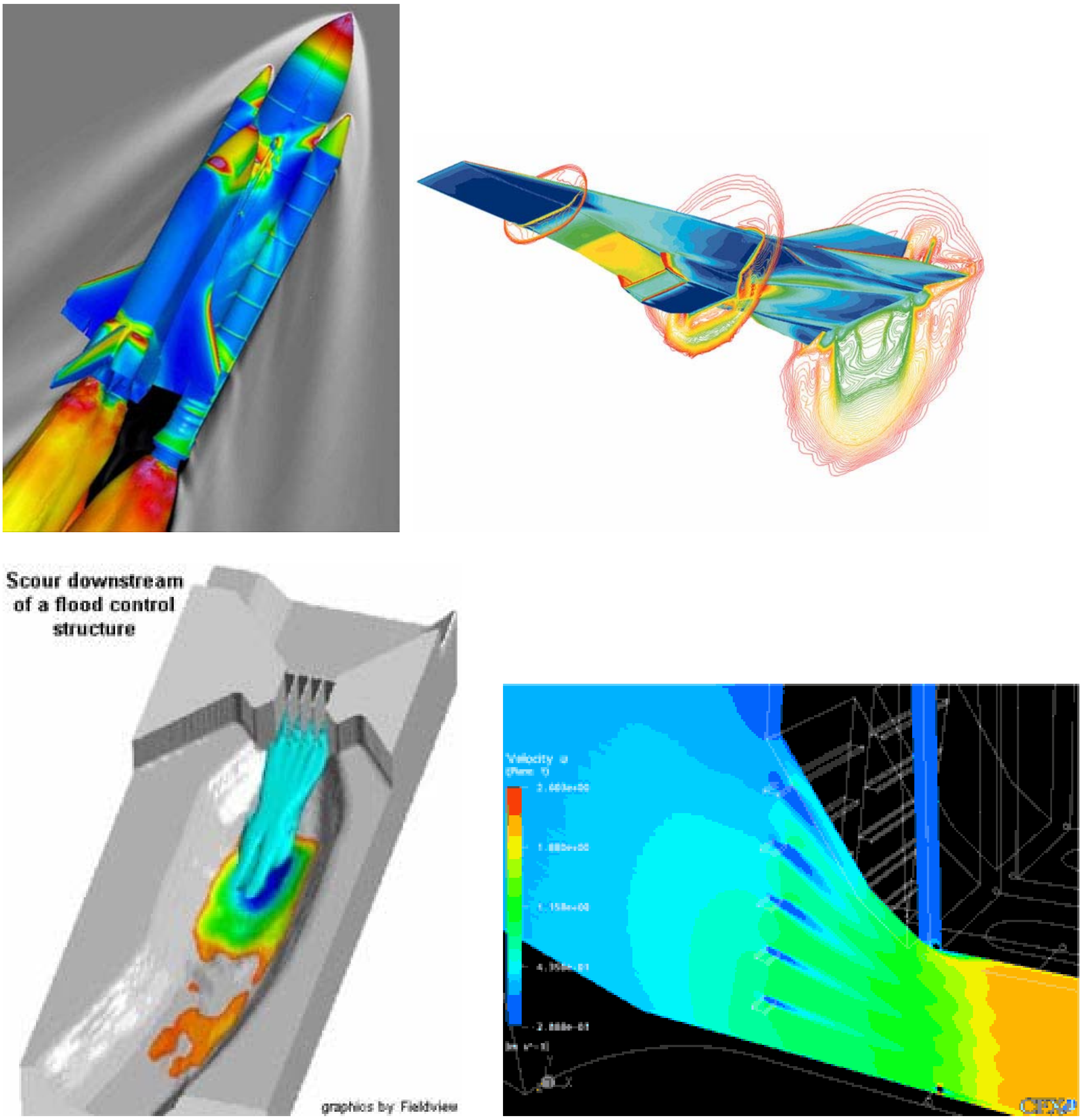


Fig. 6.18: Examples of CFD solutions

7. Basic consideration of Experimental Fluid Mechanics

7.1. Basic approach to the Dimensional Analysis

Why Dimensional Analysis?

⇒ *A significant help of Dimensional Analysis and Theory of similarity in defining a connection between the existing theory and the adequate experiment.*

⇒ *In many cases the Dimensional analysis enables adequate generalizations and formulation of generalized empiric expressions.*

⇒ *The gain of the dimensional analysis use is very important in reducing the experimental work, through transformation of certain functional relationship into relationship of dimensionless groups.*

∴ The bases of the dimensional analysis application are presented with concrete practical examples.
The presented matter is also useful for figure out the dimensional formulae and measurement units of significant physical quantities ⇒ see the table in chapter 1.2.

- **dimensional homogeneity, Rayleigh method, the significance of non-dimensional relationships and numbers,**

Equations in physics have dimensional homogeneity - not only because of their theoretical derivation but also due to the way of measurements of the physical quantities.

Definition:

All members in an equation have the same physical meaning and are expressed with same measurement units.

Example:

A form of the Bernoulli equation

$$p + \gamma h + \frac{\rho v^2}{2} = p_0 + \gamma h_0 + \frac{\rho v_0^2}{2} \tag{7-1}$$

⇒ All members have same dimensional formula - $[FL^{-2}]$ i.e. $[ML^{-1}T^{-2}]$, and are expressed with same units - $[N/m^2]$.

Another form of the Bernoulli equation:

$$\frac{p}{\rho} + gh + \frac{v^2}{2} = gH_0 \tag{7-2}$$

⇒ each member has dimensional formula $[L^2T^{-2}]$, i.e. *energy per unit mass* $[Nm/kg]$.

The most known form:

$$\frac{p}{\gamma} + h + \frac{v^2}{2g} = H_0 \tag{7-3}$$

⇒ Each member has dimensional formula [L]
⇒ *work per unit force* in [Nm/N] i.e. [m] = *hydraulic head*.

In the table *Dimensional Formulae and Measurement Units* - see chapter 1.2 ⇒ dimensional formulae for the most used physical quantities in both systems of fundamental dimensions (M,L,T,θ) (M,L,T,θ) and (F,L,T, θ) ⇒ the corresponding measurement units are given as well.

- ***Rayleigh's method***

Rayleigh's method = practical aproach to the dimensional analysis.
Vaschy' s method = general theory of dimensional analysis.

From his theoretical work in physics and experience, Lord Rayleigh made a conclusion that most of the solutions of theoretical analysis were in a form of products of powers of the involved variables and parameters:

$$P = Cq^w r^x s^y t^z \tag{7-4}$$

A simple example ⇒ the expression for *the period of the simple pendulum* with length *l* and driven by the gravity force:

$$\theta = 2\pi\sqrt{\frac{l}{g}} \tag{7-5}$$

$\therefore \quad \theta = Cl^x g^y = Cl^{1/2} g^{-1/2}$

Another example ⇒ *pressure drop per unit length* in a horizontal circular pipe, for laminar steady flow:

$$\frac{\Delta p}{L} = 32\mu \frac{v_{ave}}{d^2} \tag{7-6}$$

μ - dynamic viscosity; $v_{ave} = Q/A$ - *average velocity.*

If it is assumed that the equation (7-6) has not been discovered yet, but it is known that $\Delta p/L = f(\mu, v_{ave}, d)$, from the Rayleigh's approach - expression (7-4) ⇒

$$\frac{\Delta p}{L} = C\mu^x v_{ave}^y d^z \tag{7-7}$$

From Rayleigh's method ⇒ the exponents can be obtained as $x = 1,\ y = 1,\ z = -2$.
The constant *C* cannot be determined with this method.

In (7-4), in an arbitrarily manner, a function of four independent variables is presented.
There is no loss of generality if other number of independent variables is used - it can be any number.
However, the discussion should be based on lesser number of fundamental dimensions (M, L, T, θ).

Rayleigh's method is based on dimensional homogeneity ⇒ in the equation (7-4) both sides should be with equal dimensions.

⇒ it leads to as many algebraic equations as the number of the applied exponents.

However, it is possible: $n_{equations} = n_{exponents}$; $n_{equations} > n_{exponents}$; $n_{equations} < n_{exponents}$ ⇒ *problem?*

The concept of dependent and independent variables:

In the expression (7-4):
P - function (dependent variable; *q, r, s* and *t* - independent variables.

Same independent variables can be associated with different functions

E.g., the wall shear stress of a laminar viscous flow in a circular pipe:

$$\tau_w = 8\mu \frac{v_{sr}}{d} \tag{7-8}$$

⇒ the same independent variables as in (7-6), but with different exponents.

Which is dependent variable, and which are the independent variables, is related to the manner how the problem arises or is formulated.
If in the case of laminar flow in a pipe, the average velocity was of interest ⇒

$$v_{ave} = f(\mu, \Delta p / L, d) \tag{7-9}$$

v_{ave} - dependent variable;
μ, $\Delta p/L$, d - independent variables.

∴ Important to notice:

In the analysis of a specific problem, it is suggested not to include by mistake one or more additional dependent variable among the independent variables

∴ A method older than one century but still responds excellent to the problems of the practical dimensional analysis.

∴ *Steps of the procedure :*

- Recognition of dependent (*P*) and independent (*q,r,s,t,..*) variables in a given problem;
- Application of the general equation (7-4);
- Satisfaction of the dimensional homogeneity;
- Determination of algebraic equation, which correspond to the introduced exponents (*w,x,y,z,..*);
- Calculation of as many exponents as possible;
- Writing the final results.

The Rayleigh's method will be illustrated through examples ⇒

Simple pendulum

Assumption:

Very little is known about this phenomenon, but enough to conclude that:

$$\theta = f(g, l) \tag{7-10}$$

According (7-4) ⇒ $$\theta = C l^x g^y \tag{7-11}$$

From table *Dimensional Formulae and Measurement Units* - 1.2 ⇒ dimensional formulae:

$$[\theta] = \mathrm{L}^0\mathrm{T}^1 \quad ; \quad [l] = \mathrm{L}^1\mathrm{T}^0 \quad ; \quad [g] = \mathrm{L}^1\mathrm{T}^{-2} \tag{7-12}$$

From (7-10) and (7-12) ⇒ $L^0T^1 = (L)^x (L^yT^{-2y})$ (7-13)

$$\Rightarrow \begin{matrix} 0 = x + y \\ 1 = -2y \end{matrix} \quad \Rightarrow \quad x = 1/2 \quad \text{and} \quad y = -1/2. \qquad (7\text{-}14)$$

Finally:

$$\theta = Cl^x g^y = Cl^{1/2} g^{-1/2} \qquad (7\text{-}15)$$

The constant C cannot be determined with this method ⇒ only with experiment, or analytically.

Stokes law for fluid drag

Laminar creeping flow around a sphere is treated - as in chapter 6.4 ⇒ inertial forces can be neglected ⇒

∴ the flow is dominated be the viscous forces $F_D = f(\mu, v, d)$ ⇒

$$F_D = C\mu^x v^y d^z \qquad (7\text{-}16)$$

Dimensional formulae are:

$$[F_D] = MLT^{-2}\ ; \quad [\mu] = ML^{-1}T^{-1}\ ; \quad [v] = M^0LT^{-1}\ ; \quad [d] = M^0L^1T^0 \qquad (7\text{-}17)$$

Applying the dimensional homogeneity in (7-16) ⇒ algebraic equations:

$$\begin{matrix} x = 1 & \text{for M} \\ -x + y + z = 1 & \text{for L} \\ -x - y = -2 & \text{for T} \end{matrix} \qquad (7\text{-}18)$$

⇒ Solutions: $x = 1$, $y = 1$, $z = 1$, ⇒

$$F_D = C\mu vd \qquad (7\text{-}19)$$

In chapter 6.4, it was shown the procedure for mathematical derivation of Drag force equation from the Stokes general equation - see equation (6-30) ; $F_x = 6\pi\mu R v_\infty$.

∴ The equation (7-19), which doesn't defer from (6-30), was obtained much easier.
The constant C can be obtained with one good experiment.

Venturi flow meter

One of the classical methodologies for *flow rate measurements*.

Herschell, inspired by *Venturi's* works, invented the *Venturi flow meter*.

On *Fig.7.1* the usual form of a *Venturi meter* is given.

In chapter 5.2 the volume flow rate equation *for steady flow of inviscid incompressible fluid* was derived, see equation (5.10), i.e:

$$Q = \frac{\pi d^2}{4}\sqrt{\frac{2\Delta p}{\rho}}\left[1 - \frac{d^4}{D^4}\right]^{-1/2} \qquad (7\text{-}20)$$

If the Dimensional analysis is used ⇒

$$Q = f(\Delta p, \rho, d, D), \quad \text{i. e.} \quad Q = C\Delta p^x \rho^y d^z D^u \tag{7-21}$$

The fluid is accelerated toward the throat due to the pressure forces; with very little contribution of the viscous forces ⇒ ∴ *the viscous forces can be neglected*!

Applying the dimensional formulae ⇒

$$M^0L^3T^{-1} = M^xL^{-x}T^{-2x}M^yL^{-3y}L^zL^u \tag{7-22}$$

⇒

$$x + y = 0$$
$$-x - 3y + z + u = 3 \tag{7-23}$$
$$-2x = -1$$

⇒ $n_{ravenki} < n_{eksponenti}$ ⇒ $x = 1/2,\ y = -1/2,\ z = 2 - u$

⇒ $$Q = C\sqrt{\frac{\Delta p}{\rho}} d^2 (D/d)^u \tag{7-24}$$

∴ The expression (7-24) is similar to (7-20), but still all members are not defined.

Therefore, it is better to express *the volume flow rate as*:

$$Q = \sqrt{\frac{\Delta p}{\rho}} d^2 F(D/d) \tag{7-25}$$

∴ F(d/D) can be determined by experiment.

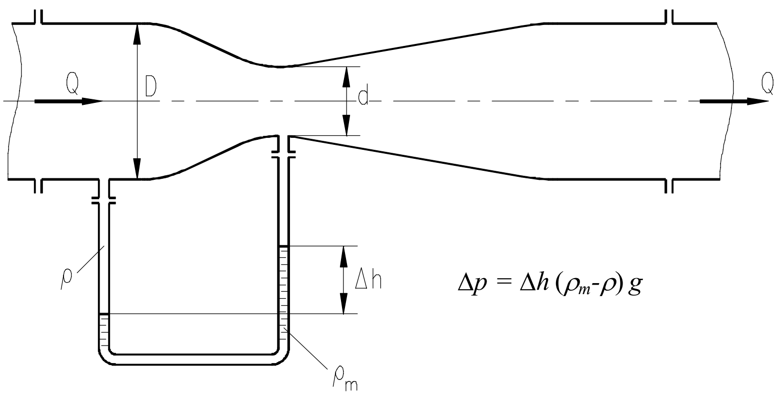


Fig. 7.1: Venturi pipe

Venturi flow meter for viscous fluid flow - refined analysis:

The viscosity of the fluid is taken into account ⇒ theoretical solution of this problem is not discovered yet.

⇒ However, the application of the dimensional analysis and corresponding experiments have given very good results.

Following the Rayleigh's approach (7-4), ⇒

$$Q = f(\Delta p, \rho, d, D, \mu) = C\Delta p^x \rho^y d^z D^u \mu^v \tag{7-26}$$

Applying the dimensional homogeneity and the corresponding dimensional formulae ⇒

$$L^3T^{-1} = C\,(ML^{-1}T^{-2})^x(ML^{-3})^y L^z L^u (ML^{-1}T^{-1})^v,$$

$$\Rightarrow \quad \begin{aligned} x + y + v &= 0 \\ -x - 3y + z + u - v &= 3 \\ -2x - v &= -1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= 1/2 - v/2 \\ y &= -1/2 - v/2 \\ z &= 2 - u - v \end{aligned}$$

$$\Rightarrow \qquad Q = C\sqrt{\frac{\Delta p}{\rho}}\, d^2 \left(\frac{D}{d}\right)^u \left(\frac{\mu}{\sqrt{\rho \Delta p}\, d}\right)^v \tag{7-29}$$

Rayleigh will define (7-29) as:

$$Q = C\sqrt{\frac{\Delta p}{\rho}}\, d^2 \cdot F_1\left(\frac{d}{D}, \rho\sqrt{\frac{\Delta p}{\rho}}\frac{d}{\mu}\right) \tag{7-30}$$

or transformed as:

$$Q = \frac{\pi d^2}{4}\sqrt{\frac{2\Delta p}{\rho}} \cdot F\left(\frac{d}{D}, \mathrm{Re}\right) = C_d \frac{\pi d^2}{4}\sqrt{\frac{2\Delta p}{\rho}} \tag{7-30a}$$

where is:

$$C_d = F\left(\frac{d}{D}, \mathrm{Re}\right) = \frac{Q}{\frac{\pi d^2}{4}\sqrt{\frac{2\Delta p}{\rho}}} = \textit{discharge coefficient} \tag{7-31}$$

$\rho\sqrt{\frac{\Delta p}{\rho}}\frac{d}{\mu} = \mathrm{Re}$ - a form of Reynolds number

∴ C_d can be obtained with experiment. Some of the experimental results for C_d for orifice meter and Venturi meter are shown on *Fig. 7.2*.

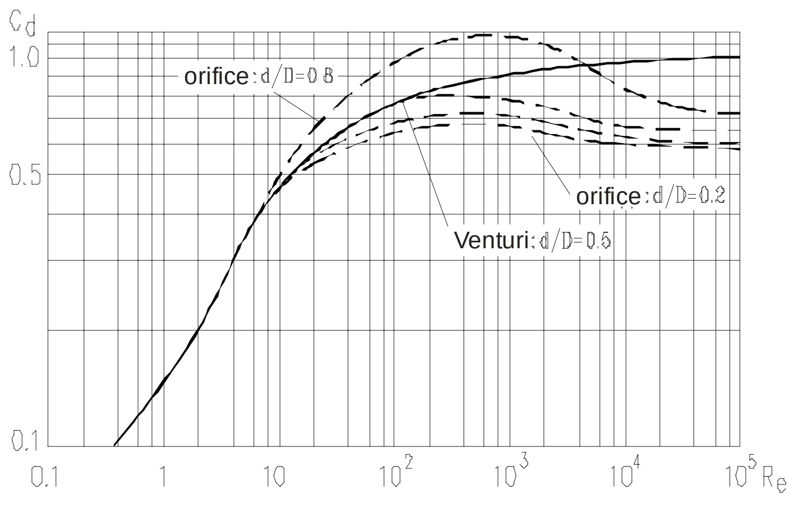


Fig. 7.2: Discharge coefficient for orifice meter and Venturi meter in function of Re

- ***Dimensionless groups***

Some of the examples lead to a transformation of certain functional dependence to another, which contains less variables.

The new variables consist of products of powers of the old ones.

For example, from the dimensional analysis of Venturi meter ⇒

$Q = f(\Delta p,\ \rho,\ d,\ D)$ (7-34) ⇒ $$\frac{Q}{d^2\sqrt{\frac{\Delta p}{\rho}}} = F(d/D)$$ (7-36)

$Q = f(\Delta p,\ \rho,\ d,\ D,\ \mu)$ (7-35) ⇒ $$\frac{Q}{d^2\sqrt{\frac{\Delta p}{\rho}}} = F(\mathrm{Re}, d/D)$$ (7-37)

The functions *F(d/D)*and *F(*Re*, d/D)* can be obtained experientially (with measurements), and sometimes analytically.

Sometimes, the functional relationships can be determined so well, that only a constant remains to be found.

Понекогаш функционалните врски можат да се определат така добро што само константата останува како непозната ⇒:

Examples ⇒ $$\frac{T}{(l/g)^{1/2}} = C; \qquad \frac{\Delta p/L}{\mu v/d^2} = C$$ (7-38)

The last two examples show the essential gain of the dimensional analysis:
∴ *The form of the function is completely determined, and only a constant remains to be found.*
⇒ In theory, only one good experiment should be enough for this.

The gain is also important in the examples (1-36) and (1-37):

- Suppose that for a function of $n = 1$ variable, 10 experimental points are necessary (*Fig. 7.3*).
- For a funcion of n=2 variables, family of 10 curves (10^2 experimental points) are needed.
- For a funcion of n variables, ⇒ ∴ 10^n experimental points.

∴ *It is obvious that, e.g. with the reduction of the expression (7-34) to an expression with only one variable (7-36), ⇒ 1000 times less experiments would be needed to determine the function!*

Besides this enormous gain in reducing the amount of experimental work, there are other advantages in applying the dimensional analysis and introducing of new variables in dimensionless forms. ⇒ *Vaschy's theorem.*

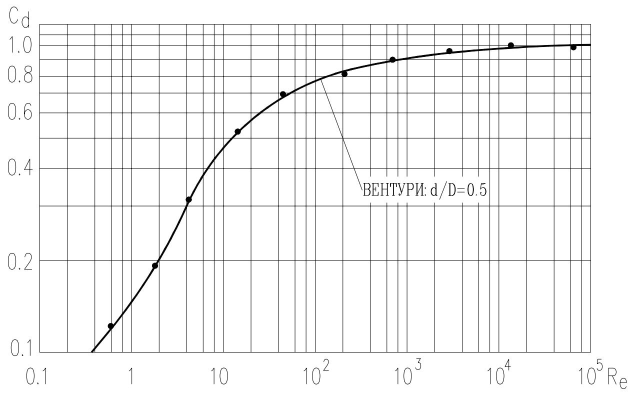


Fig. 7.3: A function of $n = 1$ *variable - 10 experimental points*

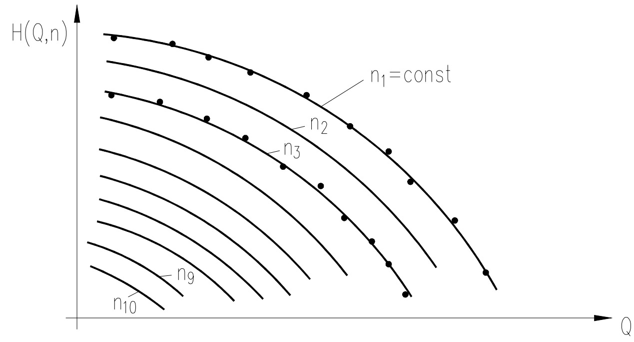


Fig. 7.4: A function of $n = 1$ *variables - 10*^{2} *experimental points*

- **Vaschy's theorem**

Any function $f(u_1, u_2, u_3, .. u_n) = 0$*, which is a relationship between "n" physical variables* u_i*, and satisfies the dimensional homogeneity, can be reduced to a function* $F(G_1, G_2, G_3, ..., G_{n-r}) = 0$ *with "n-r" dimensionless variables - where "r" is the rank of the matrix of the dimensions.*

In some literature this theorem is known as π *theorem*.

The proof of this theorem is based mostly on physical arguments - working with some physical procedures requirements and data acquisitions..

E.g., suppose that through experiments it is established a table of the values $u_1, u_2, u_3, ..., u_n$; which define the function $f(u_1, u_2, u_3, .. u_n) = 0$ in one usual system of fundamental dimensions (M,L,T for example):

u_1	u_2	u_3	u_4	...	u_i	...	u_n
u_{11}	u_{21}	u_{31}	u_{41}	...	u_{i1}	...	u_{n1}
u_{12}	u_{22}	u_{32}	u_{42}	...	u_{i2}	...	u_{n2}
...	...	...	...	...	...	...	...

in M,L,T system

⇒ *Change of the fundamental dimensions*:

Suppose that $r = 3$, and that the new dimensions are U_1, U_2, U_3 (which correspond to u_1, u_2, u_3).

⇒ The units of U_1, U_2, U_3 are identical with the values of u_1, u_2, u_3 in every row in the table.
⇒ The data in all firs three columns become one.
⇒ The other columns obtain values defined with dimensionless combinations of $u_4, u_5, u_6, ..., u_n$ with u_1, u_2, u_3:

1	1	1	u_4/u_1	...	$u_i/u_1u_2u_3$	...	u_n/u_1u_2
1	1	1	$(u_4/u_1)_1$	...	$(u_i/u_1u_2u_3)_1$	...	$(u_n/u_1u_2)_1$
1	1	1	$(u_4/u_1)_2$	...	$(u_i/u_1u_2u_3)_2$	...	$(u_n/u_1u_2)_2$
...	...	...	...	...	...	...	...

in U_1, U_2, U_3 system

To show the proof clearer, the case of fluid flow in a pipe, $f(\tau, \rho, u, d, \mu, k) = 0$, s presented.

According the previously given approach, data obtained by experiment can be systemized in a table in M,L,T system:

Example for data systematization in case of investigation of flow in pipe $f(\rho, u, d, \mu, k, \tau) = 0$					
ρ	u	d	μ	k	τ
ρ_1	u_1	d_1	μ_1	k_1	τ_1
ρ_2	u_2	d_2	μ_2	k_2	τ_2
ρ_3	u_3	d_3	μ_3	k_3	τ_3

in M,L,T system

If new dimensions *D, V, L* (corresponding to ρ, u, d),

⇒ according Vaschy, the function $f(\rho, u, d, \mu, k, \tau) = 0$ can be reduced to a funcon $F(\mu/\rho ud, k/d, \tau/\rho u^2) = 0$, and the table of the data acquisition can be simplified, i.e. the followingtable can be obtained:

Data acquisition for $f(\rho, u, d, \mu, k, \tau) = 0$ *with new dimensions D, V, L* (ρ, u, d)					
1	1	1	$\mu/\rho ud$	k/d	$\tau/\rho u^2$
1	1	1	$\mu_1/\rho_1 u_1 d_1$	k_1/d_1	$(\tau/\rho u^2)_1$
1	1	1	$\mu_2/\rho_2 u_2 d_2$	k_2/d_2	$(\tau/\rho u^2)_2$

in *D, V, L* system

Re	k	C_r
Re_1	k_1	C_{r1}
Re_2	k_2	C_{r2}
Re_3	k_3	C_{r3}

$Re = \rho ud/\mu$ - Reynolds number
k -relative roughness
C_r - resistence coefficient

The data reduction process, illustrated in the previous tables, is irreversible, but most useful for further analyses.

⇒ Example: *Ventury meter*

$$f(\rho, \Delta p, d, D, \mu, Q) = 0 \qquad (7\text{-}49)$$

Following the above approach ⇒
The matrix of dimensions has to be writen, in order to determine the rank r of the matrix of the fundamental dimensions.

∴ The matrix in M,L,T system is:

	ρ	Δp	d	D	μ	Q
M	1	1	0	0	1	0
L	-3	-1	1	1	-1	3
T	0	-2	0	0	-1	-1

(7-50)

- Since there is at least one $\det\lfloor a_{ij}\rfloor \neq 0 \Rightarrow$ the rank is $r = 3$!
- The new system of dimensions hast to be chosen. Here, D, P, L (corresponding to ρ, Δp, d) are chosen to be the new dimensions;
- The relationships M = M(D, P, L) ; L = L(D, P, L) and T = T(D, P, L) have to be found;

From $D = ML^{-3}$; $P = ML^{-1}T^{-2}$; L = L

- see *Table of dimenssional formulae* and *matrix* (7-50)

$$\Rightarrow \quad M = DL^3 ; \quad T = D^{1/2}P^{-1/2}L ; \quad L = L , \tag{7-51}$$

∴ The matrix with the new dimensionsis:

	ρ	Δp	d	D	μ	Q
D	1	0	0	0	1/2	-1/2
P	0	1	0	0	1/2	1/2
L	0	0	1	1	1	2

(7-52)

⇒ Forming of n-r dimensionless groups with ρ, Δp и d; ⇒ from $f(u_1, u_2, ..., u_n) = 0$
⇒ $F(G_1, G_2, ..., G_{n-r}) = 0$.

Here, $n = 6$, and $r = 3$; ⇒ three dimensionless groups G_1, G_2 i G_3, for which, using the dimensions of D, μ and Q in the new D, P, L system (see *matrix* (7-52)), it can be obtained:

$$G_1 = \frac{D}{d} \; ; \quad G_2 = \frac{\mu}{\rho^{1/2}\Delta p^{1/2} d} \; ; \quad G_3 = \frac{Q}{\rho^{-1/2}\Delta p^{1/2} d^2} \tag{7-53}$$

∴ The final result is:

$$F\left(\frac{Q}{d^2\sqrt{\Delta p/\rho}}, \frac{d}{D}, \mathrm{Re}\right) = 0 \tag{7-54}$$

Compare (7-54) with (7-31)!

7.2. Basic approach to the experimental investigation and application of the similarity theory - similitude

Similitude is a concept used in the testing of *engineering models*.

Engineering models are used to study complex fluid dynamics problems where calculations and computer simulations aren't reliable. Models are usually smaller than the final design, but not always. Scale models allow testing of a design prior to building, and in many cases are a critical step in the development process.

⇒ *Idea for experiments on a phenomenon in certain scale, in order to obtain data that can be converted to another scale.*

⇒ To know the relationships between the results obtained in a model phenomenon, and the results that would be obtained in a prototype phenomenon.

∴ *Galileo Galilei was among the firsts that recognized that the relationships between the model and prototype are not simple.*

⇒ *Definition of model and prototype:*

A physical model is used in various contexts to mean a physical representation of some thing. That thing may be a single item or object (for example, a bolt) or a large system (for example, the Solar System).

A prototype (or original) is an *original type*, form, or instance of some thing serving as a typical example, basis,or standard for other things of the same category.

∴ *A model is said to have similitude with the prototype (real application) if the two share geometric similarity, kinematic similarity and dynamic similarity - see Fig. 7.4..*

- *Geometric similarity* - The model is the same shape as the application, usually scaled.
- *Kinematic similarity* - Fluid flow of both the model and real application must undergo similar time rates of change motions. (fluid streamlines are similar)
- *Dynamic similarity* - Ratios of all forces acting on corresponding fluid particles and boundary surfaces in the two systems are constant.

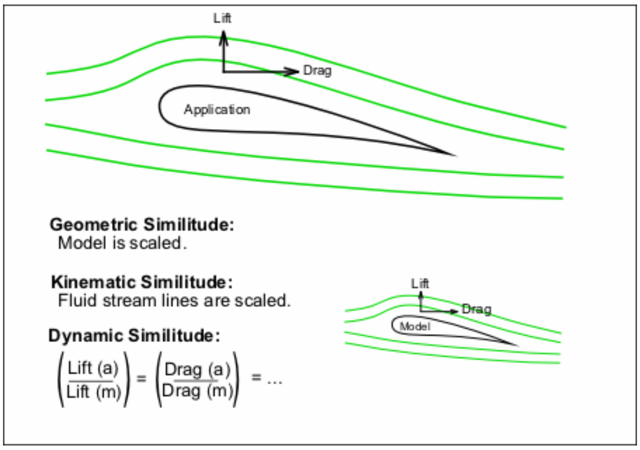


Fig. 7.5: Concept of similarity

∴ *Procedure for obtaining results from experimental analysis on certain physical model in laboratory conditions:*

- Analysis of the problem and defining of the governing equations (laws) and properties;
- Defining the similarity criteria between the model and the prototype;
- Construction of the corresponding physical model;
- Experiments on the model, measurements and data acquisition;
- Systematization and analysis of the obtained results;
- Transfer pf the obtained results from the model investigation to the prototype, using already defined similarity criteria.

∴ *The advantage of the physical model investigation is obvious!*

- **Fundamental scales of similarity:**

Following the concept of fundamental dimensions M,L,T; fundamental scales of similarity can be defined as well:

Geometric symilarity:

$$S_L = \frac{l_p}{l_m} \tag{7-55}$$

l_p - characteristic length in the prototype; l_m - characteristic length in the model.

$\therefore\ S_A = A_p / A_m = S_L^2$ - for area; $S_V = V_p / V_m = S_L^3$ - for volume; etc.

Kinematical symilarity:

$$S_v = \frac{v_p}{v_m} = S_L S_T^{-1} \tag{7-56}$$

v_p - characteristic velocity in the prototype; v_m - characteristic velocity in the model.

$\therefore\ S_a = a_p / a_m = S_L S_T^{-2}$ - for acceleration; $S_Q = S_L^2 S_v = S_L^3 S_T^{-1}$ - for volum flow rate; etc.

Material symilarity:

$$S_M = \frac{m_p}{m_m} \tag{7-57}$$

or

$$S_\rho = \frac{\rho_p}{\rho_m} = \frac{\Delta m_p / \Delta V_p}{\Delta m_m / \Delta V_m} = S_M S_V^{-1} = S_M S_L^{-3} \tag{7-58}$$

m_p - mass in the prototype; m_m - mass in the model; etc.

Dynamic symilarity:

$$S_F = \frac{F_p}{F_m} = \frac{m_p a_p}{m_m a_m} = S_M S_L S_T^{-2} \tag{7-59}$$

or

$$S_F = S_L^2 S_v^2 S_\rho \tag{7-60}$$

F_p - force in the prototype; F_m - force in the model

$\Rightarrow$

$$S_W = \frac{(\vec{F}_p, d\vec{r}_p)}{(\vec{F}_m, d\vec{r}_m)} = S_F S_L = S_L^2 S_T^{-2} S_M \text{ - for work}$$

$$S_{Ek} = \frac{\frac{1}{2} m_p v_p^2}{\frac{1}{2} m_m v_m^2} = S_M S_v^2 = S_L^2 S_T^{-2} S_M \text{ - for kinetic energy}$$

$\therefore\ S_W = S_{E_k}$

$\therefore$ *If two quantities have the same dimensional formulae, they will have the same formulae of the similarity scales* - see the *table* billow.

Scales and dimensional formulae for some physical quantities		
Scale	Quantity	Dimensional formula
S_L	*Length*	L
S_T	*Time*	T
S_M	*Mass*	M
$S_M S_L^{-3}$	*Density*	ML^{-3}
$S_L S_T^{-2} S_M$	*Force*	MLT^{-2}
$S_L^2 S_T^{-2} S_M$	*Moment of a force; Kinetic energy; Work*	ML^2T^{-2}
$S_L^{-1} S_T^{-2} S_M$	*Pressure; Shear stress; Turbulent (Reynolds) stress*	$ML^{-1}T^{-2}$
$S_L S_T^{-1} S_M$	*Impuls; Momentum*	MLT^{-1}
$S_L S_T^{-2} S_M$	*Momentum flux;*	MLT^{-2}
$S_L^2 S_T^{-1} S_M$	*Moment of momentum*	ML^2T^{-1}
$S_L^{-2} S_T^{-2} S_M$	*Pressure gradient*	$ML^{-2}T^{-2}$

- **similarity criteria for characteristic flow conditions**

Viscous forces:

Flow of incompressible fluid with linear viscous behavior (*Newtonian fluid*) is treated.

According Newton (see chapter 6.1) $\Rightarrow \tau = \mu \dfrac{dv}{dn} \Rightarrow$

$$F_\mu = \mu A \frac{dv}{dn} \tag{7-61}$$

$\therefore$ The scale of similarity for viscous forces will bee:

$$S_{F_\mu} = \frac{F_{\mu.p}}{F_{\mu.m}} = \frac{\mu_p A_p \dfrac{dv_p}{dn_p}}{\mu_m A_m \dfrac{dv_m}{dn_m}} = S_\mu S_L^2 S_T^{-1} \tag{7-62}$$

From the fundamental laws of Mechanics (D'Alembert's principle) $\Rightarrow$

Any force scale determined for a particular type of force must be equal to the determined scale of inertial force - see equation (7-59) $\Rightarrow$

$$S_{F_\mu} = S_F \qquad \Rightarrow \qquad S_\mu S_L^2 S_T^{-1} = S_M S_L S_T^{-2} \tag{7-63}$$

$\Rightarrow$ From (7-63) the scale for *dynamic viscosity* can be obtained:

$$S_{\mu} = S_M S_L^{-1} S_T^{-1} = S_{\rho} S_L S_v \tag{7-64}$$

Expressing the scales of symilarity for the separate quantities in equation (7-64) ⇒

$$\frac{S_{\rho} S_L S_v}{S_{\mu}} = \frac{\frac{\rho_p}{\rho_m}\frac{l_p}{l_m}\frac{v_p}{v_m}}{\frac{\mu_p}{\mu_m}} = 1 \tag{7-65}$$

∴ *The Reynolds criterion or law of similarity for viscous flows (in which there is an interaction between viscous and inertial forces) is derived.*

∴ *The Reynolds number of the prototype(*Re_p*) should be equal to the Reynolds number of the model(*Re_m*):*

$$\mathrm{Re}_p = \frac{\rho_p v_p l_p}{\mu_p} = \frac{\rho_m v_m l_m}{\mu_m} = \mathrm{Re}_m \tag{7-66}$$

Obviously the definition given in chapter 6.1 is valid.

$$\frac{\text{inertia force/mass}}{\text{frictional force/mass}} \propto \text{Reynolds number}$$

Gravitational forces:

$$S_G = \frac{G_p}{G_m} = \frac{\gamma_p V_p}{\gamma_m V_m} \tag{7-67}$$

G_p - gravitational force in the prototype; G_m - gravitational force in the model.

$$G = mg = \rho V g = \gamma V$$

Any force scale determined for a particular type of force must be equal to the determined scale of inertial force - see equation (7-59) ⇒

$$S_G = S_F \qquad \Rightarrow \qquad S_{\gamma} S_V = S_M S_L S_T^{-2} \tag{7-67}$$

Since: $S_V = S_L^3$, $S_M = S_L^3 S_{\rho}$ and $S_T = S_L S_{\upsilon}^{-1}$ ⇒ a form of the *Froude's criterion*:

$$\frac{S_{\upsilon}^2 S_L^{-1}}{S_{\gamma}/S_{\rho}} = 1 \tag{7-68}$$

Taking: $S_L = \frac{l_p}{l_m}$; $S_v = \frac{v_p}{v_m}$; $S_{\rho} = \frac{\rho_p}{\rho_m}$; $\gamma/\rho = g$; etc. ⇒

$$\mathrm{Fr}_p = \frac{v_p}{\sqrt{g l_p}} = \frac{v_m}{\sqrt{g l_m}} = \mathrm{Fr}_m \tag{7-69}$$

∴ *The Froude's number of the prototype (*Fr_p*) should be equal to the Froude's number of the model (*Fr_m*).*

∴ *In a similar way other similarity criteria for other accting forces can be determined. For example:*

$$M_p = \frac{v_p}{c_p} = \frac{v_m}{c_m} = M_m \quad \text{- Mach's criterion for elastic forces} \tag{7-70}$$

$c = \left(\frac{E_V}{\rho}\right)^{\frac{1}{2}}$ - acoustic velocity in the substance.

$$\mathrm{We}_p = \frac{v_p}{\sqrt{\frac{\sigma_p}{\rho_p l_p}}} = \frac{v_m}{\sqrt{\frac{\sigma_m}{\rho_m l_m}}} = \mathrm{We}_m \quad \text{- Weber's criterion for surface tension} \tag{7-71}$$

$$\mathrm{Eu}_p = \frac{p_p}{\rho_p v_p^2} = \frac{p_m}{\rho_m v_m^2} = \mathrm{Eu}_m \quad \text{- Euler's criterion} \tag{7-72}$$

Flow dominated by two forces - model and prototype in the same gravity field, and with same fluids:

Example: Very often viscous and gravitational forces have to be taken into account. ⇒

$$\mathrm{Re}_p = \mathrm{Re}_m \text{ and } \mathrm{Fr}_p = \mathrm{Fr}_m \qquad \Rightarrow \quad \frac{S_\upsilon S_L}{S_\mu / S_\rho} = 1 \text{ and } \frac{S_\upsilon^2 S_L^{-1}}{S_\gamma / S_\rho} = 1 \tag{7-73}$$

$$\Rightarrow \qquad S_L = \frac{\left(S_\mu / S_\rho\right)^{2/3}}{\left(S_\gamma / S_\rho\right)^{1/3}} \tag{7-74}$$

If the model and the prototype have to be in the same gravitational field - which is reality ⇒:

$$S_g = S_\gamma / S_\rho = 1$$

$$\Rightarrow \qquad \frac{S_v S_L}{S_\mathrm{v}} = 1 \qquad \text{and} \qquad S_v^2 S_L^{-1} = 1 \tag{7-75}$$

$S_\mathrm{v} = S_\mu / S\rho$ - scale of kinematic viscousity; $S_v = v_p / v_m$; $S_L = l_p / l_m$.

$$\Rightarrow \qquad S_L = (S_v)^{2/3} \tag{7-76}$$

∴ *S_L and S_ν are directly relate.*
If S_L is chosen, the viscosity scale S_ν would be fixed.
Example: if $S_L = 20 \Rightarrow S_\nu \approx 90$ - If the fluid of the prototype is water, it is impossible to find corresponding fluid for the model!

In practice, usually same fluids are used in the model and the prototype ⇒ $S_\mathrm{v} = S_\mu / S\rho = 1$

⇒ *Problem ⇒ To solve it, use the art of symulation:*

$\Rightarrow$ From case to case it is necessary to figure out which kind of forces is dominant $\Rightarrow$ the corresponding criterion will be used as *the basic criterion of similarity*, and the others criteria as *control criteria*!

Examples:

- If it is concluded that viscous forces are more dominant $\Rightarrow$ the Re law will be the basic one, and Fr will be used for control $\Rightarrow$

in $\dfrac{S_v S_L}{S_\nu} = 1$, for $S_\nu = S_\mu / S\rho = 1 \quad \Rightarrow \quad S_L S_v = 1 \quad \Rightarrow \quad S_v = \dfrac{v_p}{v_m} = S_L^{-1}$ (7-77)

Example: for $S_L = 50 \;\Rightarrow\; v_m = 50 v_p \;\Rightarrow$ Problem can arise for large scales!

- If it is concluded that gravitational forces are more dominant $\Rightarrow$ the Fr law will be the basic one, and Re will be used for control $\Rightarrow$

from (7-75) $\Rightarrow \quad S_v^2 S_L^{-1} = 1 \;\Rightarrow\; S_v = \dfrac{v_p}{v_m} = S_L^{1/2}$ (7-78)

$\therefore$ *No problem of using the same fluid in the prototype and model.*
Whenever is possible to use the Froude's criterion as basic criterion, and others for control.

8. Methods and examples of Applied Fluid Mechanics

In the engineering practice, very often the methods of *applied fluid mechanics* are used.

Often the *Applied fluid mechanics* is known as *Hydraulics*.

∴ *Fluid mechanics provides the theoretical foundation for hydraulics, which focuses on the engineering uses of fluid properties.*

Hydraulic topics range through most science and engineering disciplines, and cover concepts such as *pipe flow*, *dam design*, *fluid control circuitry*, *pumps*, *turbines*, *hydropowe*r, *computational fluid dynamics*, *flow measurement*, *river channel behavior and erosion*, etc.

⇒ *Some of characteristic cases of Applied Fluid Mechanics are presented in this chapter*.

8.1. Basic equations of flow in conduits and pipes

The flow of liquids and their transport through a bounded space (pipes) is treated ⇒

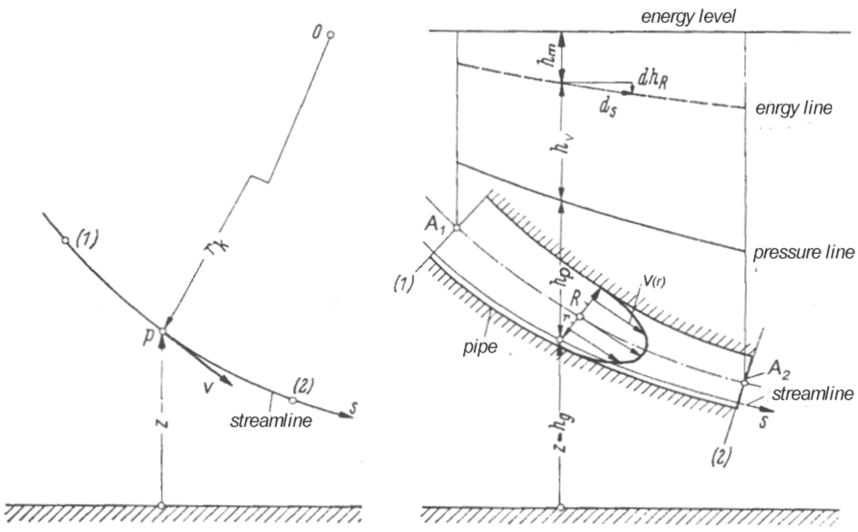
This flow corresponds to a flow through a *stream tube* (*stream filament) with defined cross-section* - see chapter 5.

∴ *The derived basic equations in chapter 5.1 can be used, if the friction due to the fluid viscosity is taken into account* ⇒

∴ The viscous friction is a cause for:

- energy losses (see energy balance on *Fig. 8.1*), and
- change of the velocity in certain cross-section (see *Fig. 8.1* and *Fig. 8.2*)

⇒ *Correction of the derived equations in chapter 5.1.*



a) *b)*

Fig. 8.1: Energy balance and energy losses

- **velocity distribution and average velocity; pressure; continuity equation; Bernoulli equation; momentum law**

- *Velocity profile, average velocity and velocity correction factors;*

Due to the viscous friction ⇒ velocity change in the pipe cross-section
⇒ *velocity profile* (see Fig. 8.2):

$v = v(r)$ - at certain radius;

$v = 0$ for $r = R$ - at the pipe walls;

$v = v_{\max}$ for $r = 0$ - at the pipe centerline.

Many of the fluid flow properties can be expressed through *the average velocity*, v_{ave}.
For example, the *Reynolds number*:

$$\mathrm{Re} = \frac{\rho v_{ave} D}{\mu} = \frac{v_{ave} D}{\nu} \qquad (8\text{-}1)$$

$D = 2R$; ν - kinematic visosity.

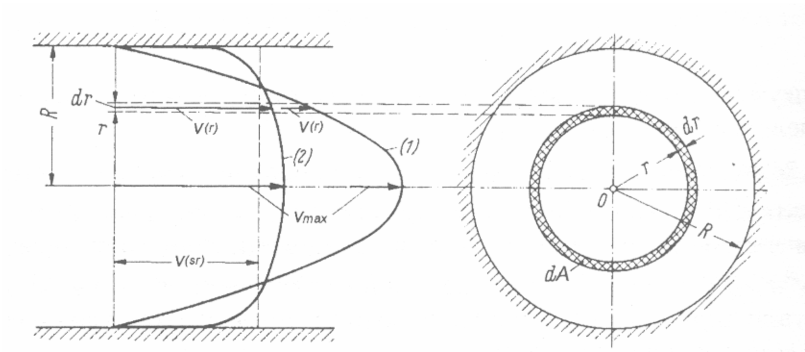


Fig. 8.2: Velocity distribution in a pipe cross-section

The average velocity can be obtained from the equation:

$$v_{ave} A = Q = \int_A v(r)dA \qquad (8\text{-}2)$$

Since, $A = R^2\pi$ and $dA = 2r\pi dr$ (infinitesimal area dA on *Fig. 8.2*) ⇒

$$v_{ave} = \frac{Q}{A} = \frac{1}{A}\int_A v(r)dA = \frac{2}{R^2}\int_0^R v(r)rdr \qquad (8\text{-}3)$$

∴ *If the profile $v(r)$ is known, the v_{ave} can be easily obtained!*

Upon an analogy derived from the averaged velocity definition (8-2), *correction factors* for some quantities can be defined:

For quantities comprising $v^2(r)$ ⇒ $$\beta = \frac{1}{v_{ave}^2 A}\int_A v^2(r)dA \qquad (8\text{-}4a)$$

For quantities comprising $v^3(r)$ ⇒ $$\alpha = \frac{1}{v_{ave}^3 A}\int_A v^3(r)dA \qquad (8\text{-}4b)$$

For *inviscid (ideal) fluid flows*: $v(r) = const \quad \Rightarrow \quad \alpha = \beta = 1$

For *viscous (real) fluid flows*: $v(r) \neq const \quad \Rightarrow \quad \alpha > 1$ and $\beta > 1$

For turbulent flows: $\alpha \approx \beta \approx 1$

For laminar flows: $\alpha > 1$ and $\beta > 1$:

According *Joseph Boussinesq*: $\beta = 1 + \lambda = 1.037$; $\alpha = 1.111$

In the engineering practice usually $\alpha \approx \beta \approx 1$ - if the flow is not characteristically laminar.

- *Continuity equation*

Following the average velocity concept, the continuity equation for incompressible fluid flow (5-4), as derived in chapter 5.1, is valid in this case as well. ⇒:

$$Q = \int_{A_1} v dA = \int_{A_2} v dA = v_1 A_1 = v_2 A_2 = vA = const \tag{8-5}$$

According *Fig. 8.1*, ⇒ v_1, A_1 = velocity and area at the cross-section (1); v_2, A_2 = velocity and area at the cross-section (2).

- *Bernoulli's equation*

If the average velocities, in sections (1) and (2) on *Fig. 8.1*, are v_1 and v_2; for *steady viscous incompressible fluid flow*, the energy losses have to be taken into account ⇒ the equation (4-25) has to be transformed (see also chapter 5.1) ⇒

$$\frac{v_1^2}{2g} + z_1 + \frac{p_1}{\gamma} = \frac{v_2^2}{2g} + z_2 + \frac{p_2}{\gamma} + h_m \tag{8-6}$$

h_m - *specific energy loss* between section (1) and (2) = *head loss* in $\left[\frac{\text{Nm}}{\text{N}}\right]$ or $[\text{m}]$ -.see *Fig. 8.1*,

Bernoulli's energy equation can be expressed in Nm/kg as:

$$\frac{v_1^2}{2} + gz_1 + \frac{p_1}{\rho} = \frac{v_2^2}{g} + gz_2 + \frac{p_2}{\rho} + E_m \tag{8-7}$$

$$E_1 = E_2 + E_w \tag{8-7a}$$

$E_w = \Delta E_w = gh_m$ - *specific energy losses* between (1) and (2) in $\frac{\text{Nm}}{\text{kg}}$.

For *unsteady flow*, the time dependent member has to be added:

$$\frac{v_1^2}{2g} + z_1 + \frac{p_1}{\gamma} = \frac{v_2^2}{2g} + z_2 + \frac{p_2}{\gamma} + \frac{1}{g}\int_{(1)}^{(2)} \frac{\partial v}{\partial t} ds + h_m \tag{8-8}$$

For strongly *laminar flow*, the correction factors (8-4) have to be taken into account ⇒

$$\alpha_1 \frac{v_1^2}{2g} + z_1 + \frac{p_1}{\gamma} = \alpha_2 \frac{v_2^2}{2g} + z_2 + \frac{p_2}{\gamma} + \frac{1}{g}\int_{(1)}^{(2)} \beta \frac{\partial v}{\partial t} ds + h_m \tag{8-9}$$

In the engineering practice usually $\alpha \approx \beta \approx 1$, usually the equations (8-6) and (8-8) are used.

- *Momentum law*

The equation (5-38), $\vec{F}_r = -\left(p_1 + \rho v_1^2\right)\vec{A}_1 - \left(p_2 + \rho v_2^2\right)\vec{A}_2 + \vec{G}_{1-2}$, derived chapter 5.4, is transformed following the previously given definitions:

For: $\vec{G}_{1-2} = 0$ - no body forces; $p = const$ in the corresponding cross-section;
$z = 0$ - horizontal plane; and streight pipe $\Rightarrow$ $A_2 \to +dA_2$; $\vec{A}_1 \to -dA_1$ $\Rightarrow$

$$\int_{A_1}\left(p + \rho v^2\right)dA_1 - \int_{A_2}\left(p + \rho v^2\right)dA_2 = F_r \tag{8-10}$$

From the correction factor definition (8-4a) $\Rightarrow$

$$\int_A\left(p + \rho v^2\right)dA = \left(p + \beta\rho v_{ave}^2\right)A$$

$$\Rightarrow \quad \left(p_1 + \beta_1\rho v_1^2\right)A_1 - \left(p_2 + \beta_2\rho v_2^2\right)A_2 = F_r = -F_{1-2} \tag{8-11}$$

v_1 and v_2 are average velocities in cross-sections (1) and (2).
In the engineering practice usually $\alpha \approx \beta \approx 1$, than the equations derived in chapter 5 are valid here as well.

- **energy losses - linear and local losses**

As explained previously in this chapter, the energy losses are expressed through the *head loss* h_m - see *Fig. 8.1* and equation (8-6).
In general, head losses are caused by all *resistances to the flow*.

The flow resistances in conduits and pipes can be classified in *three groups*:

- flow resistance due to the friction in the straight pipe part - *linear head losses;*
- local flow resistances - *local head losses*;
- losses due to the hydraulic machine (pump/turbine/motor) built in the pipeline - *built in hydraulic machine head losses*.

  - *Linear head losses*

The linear head losses are caused by the friction forces in the fluid flow. They can be express by the pressure drop $\Delta p = p_1 - p_2$ between the observed cross-sections (1) and (2) (see *Fig. 8.1*):

$$\Delta p = \xi_f \frac{l}{D}\frac{\rho v^2}{2} \tag{8-12}$$

The equation (8-12) is obtained by use of *Dimensional Analysis* and *Theory of similarity*.
$v = v_{ave} = Q / A$ - average velocity;
l and D - pipe length and diameter between the observed cross-sections;
$\xi_f = \lambda$ - *pipe friction factor*, usually obtained experimentally.

$$\xi_f = \lambda = f(\mathrm{Re}, k / D) \tag{8-13}$$

k/D - relative roughness of the pipe wall.

The head loss h_m in $\frac{\text{Nm}}{\text{N}}$ (or m *l. c.*) can be expressed with the equation - *Darcy's formula*:

$$h_m = \frac{\Delta p}{\gamma} = \frac{\Delta p}{\rho g} = \lambda \frac{l}{D}\frac{v^2}{2g} \tag{8-14}$$

The *hydraulic gradient* (drop) , or pressure drop per unit length is defined as:

$$I = \frac{dh_m}{ds} = \frac{d(\Delta p/\gamma)}{ds} = \lambda \frac{v^2}{2gD} \tag{8-15}$$

For a pipe with $D \neq const$, i.e. $D = D(s)$, h_m can be obtained by integration of (8-15) $\Rightarrow$

$$h_m = \frac{1}{2g}\int_{(1)}^{(2)} \frac{\lambda}{D} v^2 ds \tag{8-15a}$$

If the cross-sections changes (D_i) are on separate pipe parts (l_i), the total linear head loss can be obtained as sum of the separate linear head losses:

$$h_m = \frac{1}{2g}\sum_{i=1}^{i=n} \lambda_i \frac{l_i}{D_i} v_i^2 \tag{8-16}$$

As shown on *Fig. 8.3* and *Fig. 8.3A*, the velocity profile changes, from the pipe entrance to certain length $L_0 = l_E$, after which the flow is fully developed (stabilized) fluid flow $\Rightarrow$ after L_0 the profile doesn't change $\Rightarrow$ $\partial v/\partial y = const$.

The length of entrance $L_0 = l_E$, can be obtained according Boussinesq, with the folloing expressions:

$L_0 = 3.84\frac{D}{\lambda}$ - for laminar flow; $L_0 = 0.52\frac{D}{\lambda}$ - for turbulent flow (8-17)

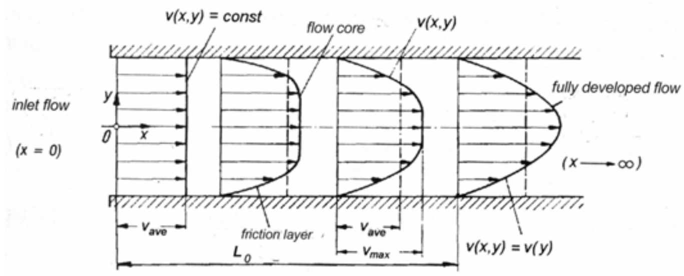


Fig 8.3: Length of flow development

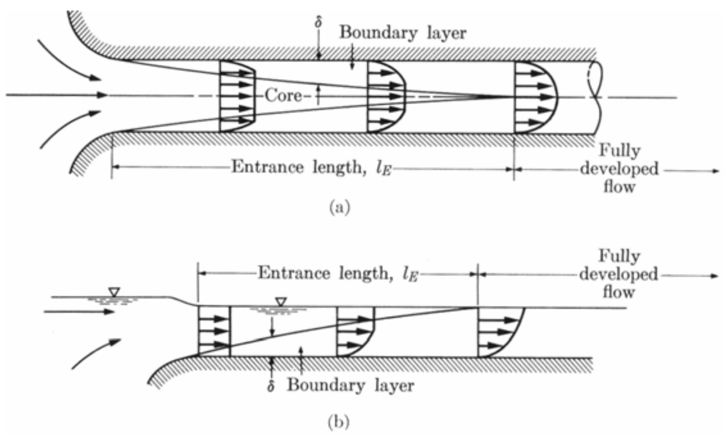


Fig. 8.3A: Developmet of uniform boudary layers: a)circular tube; b) 2-D open channel

A fully developed fluid flow is considered in this chapter.

If the flow is fully developed, since $\partial v / \partial y = const$, the *shear stress* (*Fig. 8.4*) will be also constant along the entire pipe length ⇒:

$$\tau_w = \mu \left(\frac{\partial v}{\partial y} \right)_{y=R} = const \tag{8-18}$$

∴ The *friction force* will be:

$$F_w = \tau_w O l = \tau_w A_0 \tag{8-19}$$

O - wetted perimeter of the pipe cross-section;
$A_0 = Ol$ - wetted area of the pipe walls on a distance L - between sections (1) and (2) - *Fig.8.4*.

The pressure force between (1) and (2) is:

$$F_p = (p_1 - p_2) A \tag{8-20}$$

$A = \pi D^2 / 4$ - area of the pipe cross-section.

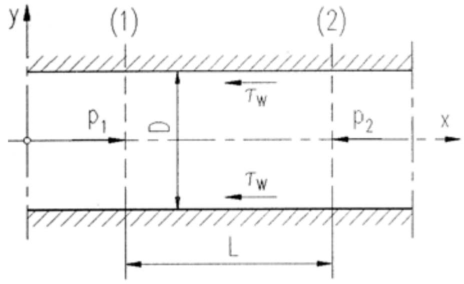


Fig. 8.4: Shear stress at pipe wall

Summing all forces in the flow direction ⇒

$$(p_1 - p_2) A = \Delta p A = \tau_w O l$$

⇒ head loss $h_m = \Delta p / \gamma$:

$$h_m = \frac{\Delta p}{\gamma} = \tau_w \frac{Ol}{A \rho g} \tag{8-21}$$

$$\therefore \; h_m = f\left(\frac{A}{O} \right) = f(R_h)$$

Hydraulic radius: $$R_h = \frac{A}{O} \tag{8-22}$$

A - area of the fluid flow cross-section; O - wetted perimeter.

Some examples given on *Fig.* 8.5.

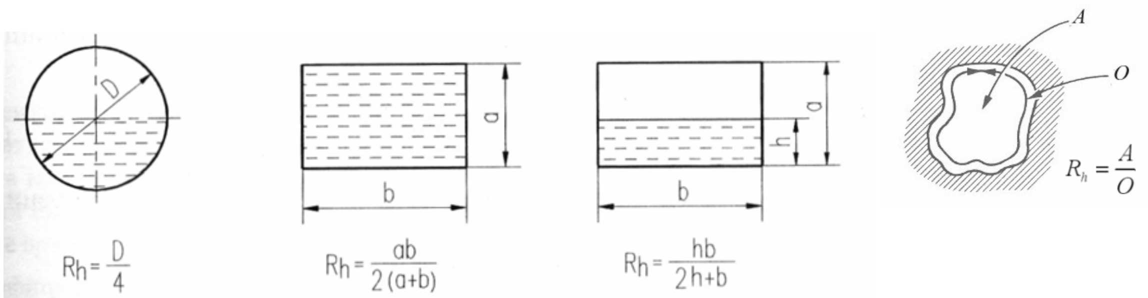


Fig. 8.5:Conduit cross-sections and hydraulic radius

For circular pipe, with a fully filled out cross-section ⇒ $A = \pi D^2 / 4$; $O = D\pi$ ⇒

$$R_h = \frac{\pi D^2 / 4}{\pi D} = \frac{D}{4} \quad (8\text{-}23)$$

⇒ *hydraulic diameter*:

$$D = D_h = 4R_h = \frac{4A}{O} \quad (8\text{-}23a)$$

∴ The *Darcy's formula* can be transformed into:

$$h_m = \lambda' \frac{l}{4R_h} \frac{v^2}{2g} = \lambda' \frac{l}{D_h} \frac{v^2}{2g} \quad (8\text{-}24)$$

∴ *The equation (8-24) can be used for linear head loss for flow in conduits with any shape cross-sections* - noncircular sections as well (see *Fig. 8,5*).

$\lambda' = G\lambda$ - *corrected friction factor*; the coefficient G is different for different cross sections.

$\lambda' \approx \lambda$ - for turbulent flow;

$\lambda' = (0.4 \div 1.5)\lambda$ - for laminar flow

∴ *The Chezy formula for the average velocity over the flow section* can be determined from the equation (8-24):

$$v = v_{ave} = \sqrt{\frac{8g}{\lambda}} \sqrt{R_h I} = C\sqrt{R_h I} \quad (8\text{-}25)$$

$I = h_m / l$ - *hydraulic gradient*, defined according the equation (8-15)

$$C = \sqrt{8g/\lambda} = f(v, \rho, \mu, k, \text{channel size, chanel shape}) \text{ - \textit{flow resistance factor}} \quad (8\text{-}26)$$

$$\Rightarrow \qquad \lambda = \frac{8g}{C^2} \quad (8\text{-}26a)$$

Manning obtained experimentally the following equation;

$$C = \frac{1}{n} R_h^{1/6} = \frac{1}{n}\left(\frac{D_h}{4}\right)^{1/6} \quad (8\text{-}26b)$$

$$\Rightarrow \qquad \lambda = \frac{8g}{C^2} = \frac{8gn^2}{R_h^{1/3}} \quad (8\text{-}26c)$$

n - *Manning's roughness coefficient*; depends of the conduit roughness - see the *Literature*.

$n = 0.040$ - for very rough surfaces; $n = 0.009$ - for smooth pipes.

For circular pipes with normal roughness $n \approx 0.012 \Rightarrow \lambda \approx 0.0179 D^{-1/3}$

The Chezy formula is especially used for hydraulic computations of open channel flows (se *Fig. 8*.6) ⇒ more details in chapter 8.4.

The hydraulic gradient in this case (Fig. 8.6) can be easilly expressed as:

$$I = \frac{h_m}{l} = \frac{(p_1 - p_2)/\gamma}{l} = \frac{h_1 - h_2}{l} = \sin\alpha \quad (8\text{-}27)$$

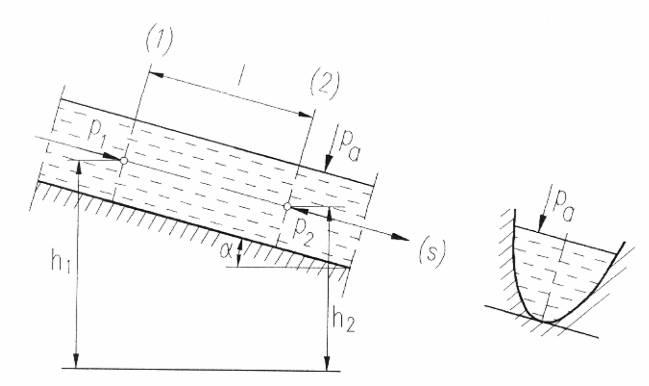


Fig. 8.6: Oppen channel - hydraulic gradient

- *Local head losses:*

Local head losses = local flow resistances that appear in nonuniform flows in conduits, as:

- *Increase or decrease in fluid velocity and pressure* - e.g., change of the size or the shape of the conduit cross section (pipe diameter for example, or inflow in a reservoir);
- *Built in of metering devices* - e.g., Venturi meter;
- *Flow control devices* - e.g., valves, hydraulic components of automatic control, etc;
- *Change in flow direction* - e.g., elbows etc;
- *Flow around immersed objects* - e.g., flows in heat exchangers, porous media flows, multiphase flows etc.

$\therefore$ *Adequate energy dissipation or local head loss*

The local head losses can be express by the pressure drop Δp due to the local resistance, or as a corresponding head loss h_m :

$$h_m = \frac{\Delta p}{\gamma} = \frac{\Delta p}{\rho g} = \xi \frac{v^2}{2g} \tag{8-28}$$

$v = v_{ave} = Q / A$ - average velocity in the uniform flow region;
ξ - *local head loss coefficient*;

$$\xi = f(\text{geometry}, \text{Re}) \tag{8-29}$$

$\Rightarrow$ ξ is experimentally obtained $\Rightarrow$ *see corresponding values for different local resistances in the literature.*

Usually there are several local resistances in a hydraulic conduit system (pipeline for example).
In that case, the *total local head loss* will be:

$$h_m = \sum_1^m \xi_i \frac{v_i^2}{2g} = \frac{1}{2g} \sum_{i=1}^m \xi_i v_i^2 \tag{8-30}$$

m - number of the local resistances

Examples for solving problems concerning energy losses - linear and local - will be presented on the tutorials classes.

- *Losses due to a built in hydraulic machine:*

The head loss due to built in hydraulic machine (pump/turbine/motor) can be defined as:

$$h_M = \frac{N_M}{\rho g Q} \tag{8-31}$$

N_M - *hydraulic power of a hydraulic machine.* $\Rightarrow$ see also equation (5-50) in chapter 5.4.

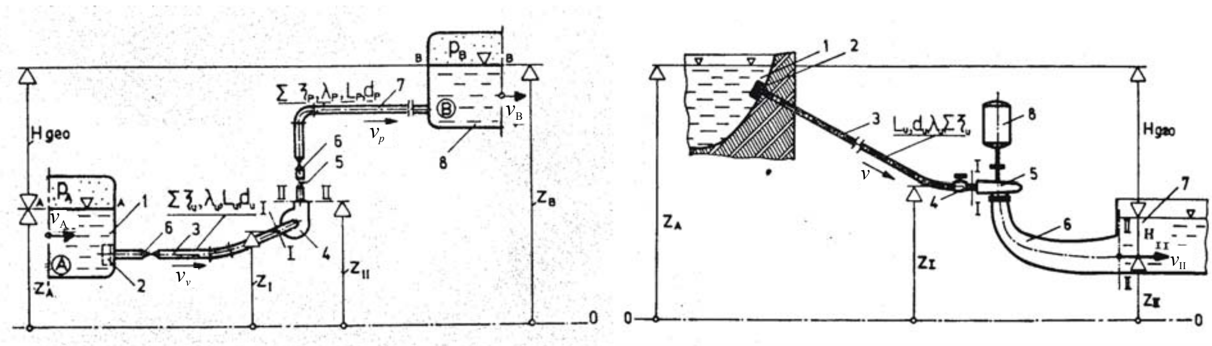
$$N_M = \frac{N_T}{\eta_T} > 0 \quad \text{- for turbine/motor;} \qquad N_M = N_p \eta_p \quad \text{- for pump} \tag{8-32}$$

N_T - power delivered to the turbine shaft; N_P power delivered from a motor to the pump shaft.

$$\therefore \quad h_M > 0 \text{ - for a built in turbine;} \qquad h_M < 0 \text{ - for a built in pump} \tag{8-33}$$

$\therefore$ *The total head loss in a pipeline with "n" partial pipe parts an "m" local resistances in the pipeline with built in hydraulic machine will be* - see *Fig. 8.7:*

$$h_m = \frac{1}{2g}\left(\sum_{i=1}^{i=n} \lambda_i \frac{l_i}{D_i} v_i^2 + \sum_{i=1}^{i=m} \xi_j v_j^2\right) + \frac{N_M}{\rho g Q} \tag{8-34}$$



a) built in pump *b) built in turbine*

Fig. 8.7: Schemes of pipelines with built in hydraulic machine

8.2. Laminar and turbulent incompressible flows in pipes

- **velocity profiles for laminar flow - velocity and friction laws**

Steady laminar flow in a circular pipe with uniform cross-section ($D = const$) is treated.

According chapter 6.3 (*Fig. 6.7*) $\Rightarrow$ velocity profile for steady laminar flow in a pipe, as on *Fig. 8.8* (see also *Fig. 8.2* in this chapter):

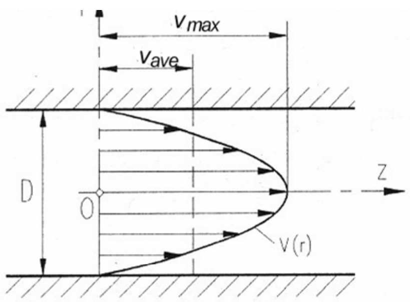


Fig. 8-8: Steady laminar flow in a circular tube of a constant diameter

$$v = v(r) = \frac{1}{4\mu}\left(-\frac{dp}{dz}\right)\left[\left(\frac{D}{2}\right)^2 - r^2\right] \tag{8-35}$$

Since in a cross-section (along the normal) $\Rightarrow -dp/dz = const \Rightarrow$

$$Q = \int_{r=0}^{D/2} v(r) 2r\pi dr = \frac{\pi D^4}{128\mu}\left(-\frac{dp}{dz}\right) \tag{8-36}$$

$\therefore$ *average velocity*:

$$v_{ave} = \frac{Q}{A} = \frac{1}{A}\int_{r=0}^{D/2} v(r) 2r\pi dr = \frac{D^2}{32\mu}\left(-\frac{dp}{dz}\right) \tag{8-37}$$

According equation (6-26), $v_{ave} = \frac{1}{2} v_{\max}$

$\therefore$ *The pressure distribution along the pipe* (equation (6-27) in chapter 6.3) $\Rightarrow$

$$\Delta p = p_1 - p_2 = \frac{32\mu L v_{ave}}{D^2} \tag{8-38}$$

$\therefore$ *Head loss, obtained theoretically*:

$$h_m = \frac{\Delta p}{\rho g} = 64 \frac{L}{D}\frac{v_{ave}^2}{2g}\frac{\mu}{\rho v_{ave} D} = \frac{64}{\text{Re}}\frac{L}{D}\frac{v_{ave}^2}{2g} \tag{8-39}$$

Where is: L - pipe length between two secctions (1) and (2); $A = \pi D^2/4$; $\text{Re} = \frac{\rho v_{ave} D}{\mu} = \frac{v_{ave} D}{\nu}$

$\text{Re} < 2320$ for laminar flow.

Comparing with *Darcy's formula* (8-14), $h_m = \lambda \frac{l}{D}\frac{v^2}{2g}$, $\Rightarrow$

Pipe friction factor for laminar flow:

$$\lambda = \frac{64}{\text{Re}} \tag{8-40}$$

According equation (8-18) and velocity distribution (8-35)
$\Rightarrow$ *the shear stre*ss at any point in a cross-section:

$$\tau = \mu \frac{dv}{dr} = \frac{1}{2}\frac{dp}{dz} r \tag{8-41a}$$

$\Rightarrow$ *maximum shear stress at pipe wall:*

$$\tau_w = \frac{1}{2}\frac{dp}{dz} R \quad \text{and} \quad \Rightarrow \quad \frac{\tau}{\tau_w} = \frac{r}{R} \tag{8-41b}$$

Taking (8-37) into account $\Rightarrow$:

$$\tau_w = -\frac{4\mu}{R} v_{ave} \tag{8-41c}$$

Using the velocity profile equation (8-35) $\Rightarrow$ *the correction factors* α *and* β (equations (8-4)) can be easily obtained:

$$\alpha = 2.0 \quad \text{and} \quad \beta = 1.33$$

$\therefore$ *Laminar flow in pipes is very rare.*
$\therefore$ *Most of the flows in pipes are turbulent.*

- **velocity profiles for turbulent flow, velocity and friction laws, roughness effects**

Most of the flows in pipes are turbulent.

Data based mostly on experiments $\Rightarrow$ *semi-empiric methods and experiments.*

CFD is playing a significant role it this field as well.

- *Velocity profiles for turbulent flow in pipes*

For friction factor determination, the velocity profile is important!

Profiles of mean velocity $\bar{v}$ are considered. $v = \bar{v} + v'$, see expressions (6-44).

Several empirical formulae are obtained, mainly based on *Prandtl* and *Karman* theories.

The following conclusion can be derived from the performed experiments and obtained expressions:

$\therefore$ *The velocity profile in turbulent flow in pipes varies with the Reynolds number!* - see *Fig. 8.10.*

$\therefore$ *Experiments show that, with respect to the nonuniform boundary layers, it is possible to represent the pipe-velocity profiles by a power law* - see *Fig. 8.10.* $\Rightarrow$:

$$\frac{\bar{v}_x}{\bar{v}_{x\max}} = \left(\frac{y}{R}\right)^n = \left(1 - \frac{r}{R}\right)^n \tag{8-42}$$

$\Rightarrow$ the exponent $n = f(\mathrm{Re})$.

From the *Nikuradse* experiments (see also $\Rightarrow$ *Fig. 8.10*):

Re	4×10^3	2.3×10^4	1.1×10^5	1.1×10^6	2×10^6	3.2×10^6
$1/n$	6.0	6.6	7.0	8.8	10	10
$\bar{v}_{xave}/\bar{v}_{x\max}$	0.791	0.806	0.817	0.853	0.865	0.865

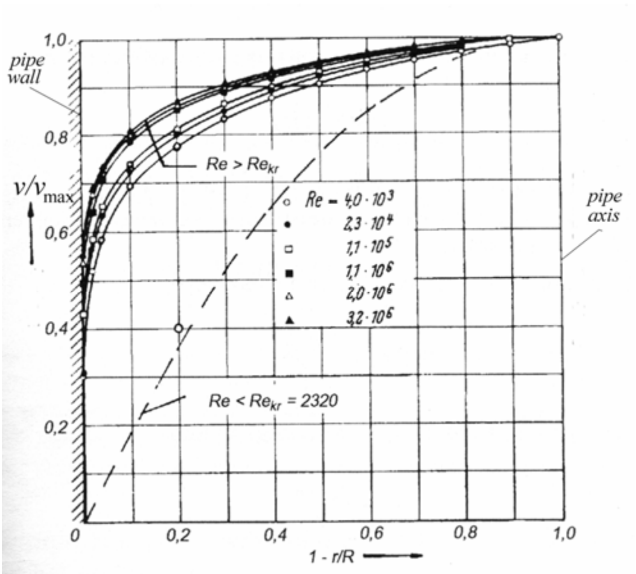


Fig. 8.10: Velocity profiles in pipe according Nikuradze

From the equation (8-42) and the average velocity definition (8-3, *the ratio of the average velo*city can be obtained: ⇒

$$\frac{\bar{v}_{ave}}{v_{\max}} = \frac{2}{(1+n)(2+n)} \tag{8-43}$$

For most common case: $n = 1/7$ ⇒ $\frac{\bar{v}_x}{\bar{v}_{x\max}} = \left(\frac{y}{R}\right)^{1/7} = \left(1 - \frac{r}{R}\right)^{1/7}$; $\frac{\bar{v}_{ave}}{v_{\max}} = 0.817$

- *Friction factor for turbulent flow in circular smooth pipes*

According the equation (8-21) and the Darcy's formula (8-14) for a circular smooth pipe ⇒

⇒ *head loss,* $$h_m = \frac{\Delta p}{\gamma} = \tau_w \frac{OL}{A\rho g} = \tau_w \frac{4L}{D\rho g} = \lambda \frac{L}{D}\frac{v_{ave}^2}{2g} \tag{8-44}$$

⇒ *pressure drop,* $$\Delta p = \tau_w \frac{L}{D/4} \tag{8-45}$$

Where: $v_{ave} = \bar{v}_{ave}$ - *average mean velocity.*

From equation (8-44), the following quantities can be derived:

pipe wall shear stress $$\tau_w = \lambda \frac{\rho v_{ave}^2}{8} \tag{8-46}$$

shear velocity $$v_\tau = \sqrt{\tau_w / \rho} \tag{8-47}$$

The *shear velocity* is defined in the *Prandtl turbulent boundary layer theory* (see also chapter 6).

From (8-46) and (8-47) ⇒

$$v_\tau = v_{ave}\sqrt{\frac{\lambda}{8}} \tag{8-48}$$

$$\frac{v_{ave}}{v_\tau} = \sqrt{\frac{8}{\lambda}} \tag{8-49}$$

From the *Prandtl's theory for turbulent flows* and *Nikuradze's experiments*, also ⇒

$$\frac{v_x}{v_\tau} = 5.75\log\left(\frac{v_\tau y}{\nu}\right) + 5.5 \tag{8-50}$$

for the circular pipe axis $$\frac{v_{x\max}}{v_\tau} = 5.75\log\left(\frac{v_\tau R}{\nu}\right) + 5.5 \tag{8-51}$$

Where: $\nu = \frac{\mu}{\rho}$; $v_x = \bar{v}_x$. $v_{x\max}$ - velocity at the pipe centerline ($y = R$).

Taking into account the equation (8-49), the equation (8-51) can be transformed, and ⇒

∴ *Friction law for smooth pipes:*

$$\frac{1}{\sqrt{\lambda}} = A\log\left(\mathrm{Re}\sqrt{\lambda}\right) + B \tag{8-52}$$

$\mathrm{Re} = \frac{v_{ave}D}{\nu}$ - Reynolds number; A and B - constants, which can be experimentally obtained.

Experiments of many researchers show that $A = 2.0$ and $B = -0.8$ $\Rightarrow$

$$\frac{1}{\sqrt{\lambda}} = 2.0\log\left(\text{Re}\sqrt{\lambda}\right) - 0.8 \tag{8-52a}$$

However, for *different regimes of flow in smooth pipes*, expressions for certain values of Re are derived:

For laminar flow $\text{Re} < 2329$, the equation (8-40) is valid:

$$\lambda = \frac{64}{\text{Re}} \tag{8-53}$$

For flows with $2000 < \text{Re} < 10^5$, Blasius derived an empirical expression:

$$\lambda = 0.316/(\text{Re})^{1/4} \tag{8-54}$$

$\therefore$ $\lambda = f(\text{Re})$ *for flows in smooth pipes*

- *Roughness effects - friction factor for turbulent flow in rough circular pipes*

The general funcional dependence (8-13) has to be concidered:

$$\lambda = f(\text{Re}, k_s / D) \tag{8-55}$$

k_s - sand grain roughness (*absolute rougness*)

However, for "*fully rough*" conditions, using the previous approach, the experiments have shown that Re has a very little influence, and the following dependence can be derived:

$$\frac{1}{\sqrt{\lambda}} = C\log\left(R/k_s\right) + E \tag{8-56}$$

C and E - constants, which can be experimentally obtained.

Nikuradze, with his experiments for *fully developed rough flow*, derived the equation:

$$\frac{1}{\sqrt{\lambda}} = -2\log\left(k_s/D\right) + 1.14 \tag{8-56a}$$

Where, $D = 2R$

For the *transition zone*, between smooth and fully rough conditions, obviously $\lambda = f(\text{Re}, k_s / D)$, and the Colebrook-White semi-empirical formula gives acceptable results:

$$\frac{1}{\sqrt{\lambda}} + 2\log\left(\frac{k_s}{D}\right) = 1.14 - 2\log\left(1 + 9.35\frac{D/k_s}{\text{Re}\sqrt{\lambda}}\right) \tag{8-57}$$

$\therefore$ Using the preceding results for *smooth*, *rough*, and *smooth-to-rough transition factors*, *Moody* developed a general resistance diagram for uniform flow in conduits. A form of the Moody's diagram (which has been widely used) is presented on *Fig. 8.11*.

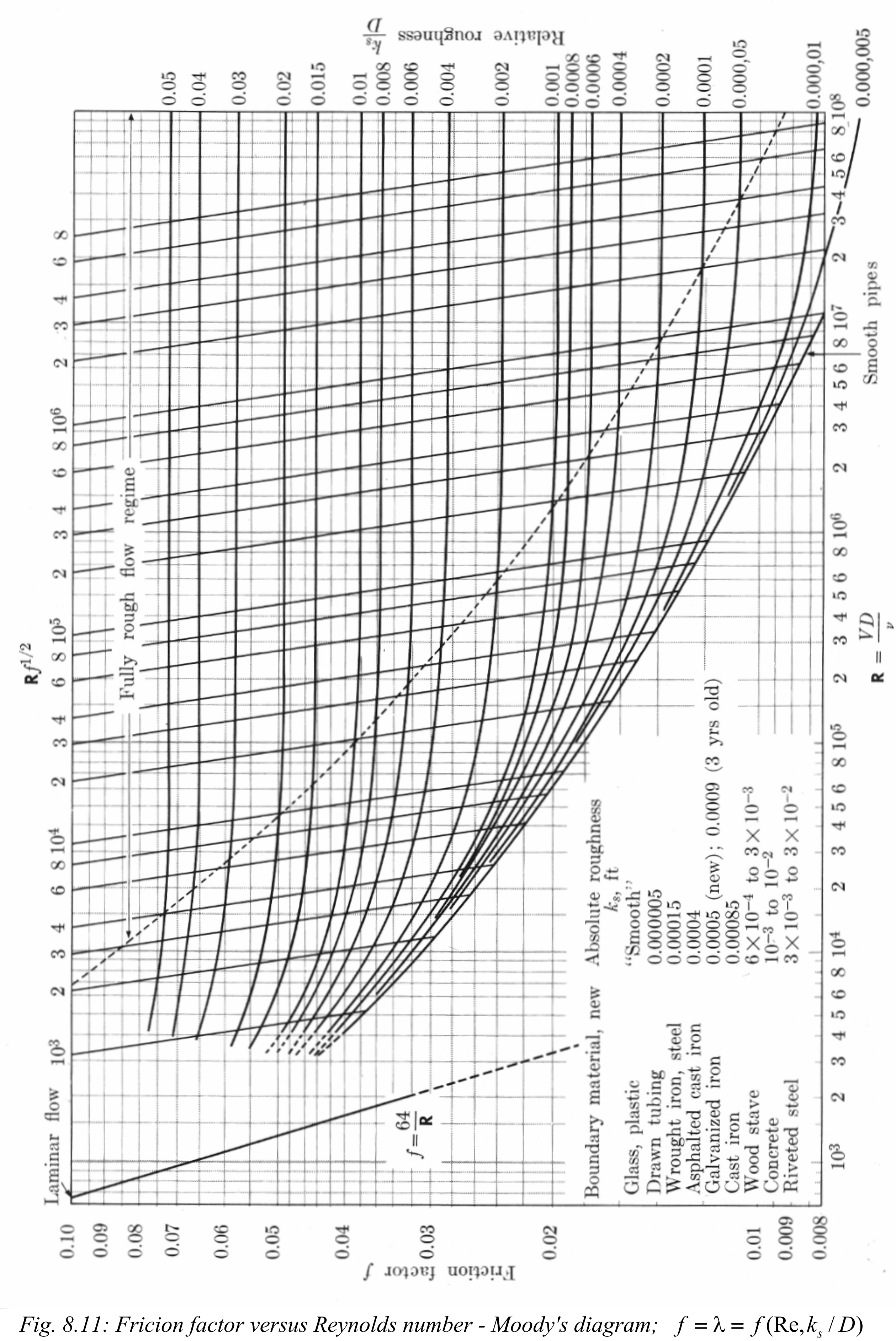


Fig. 8.11: Fricion factor versus Reynolds number - Moody's diagram; $f = \lambda = f(\mathrm{Re}, k_s / D)$

- **examples for pipe-flow computation**

The computation of steady flow of constant-density fluids flow through pipes involves the simultaniuos solutions of the two equations:

Continuity equation: $Q = VA = v_{ave} A$

Darcy' formula: $h_m = \dfrac{\Delta p}{\rho g} = \lambda \dfrac{L}{D} \dfrac{V^2}{2g}$

Where: $\mathrm{Re} = \dfrac{VD}{\nu}$; $\lambda = f = f(\mathrm{Re}, k/D)$ - obtained from the diyagram on *Fig. 8.11*

There are three basic problems, namely:

(a) *Head loss* $\Rightarrow$ for given: *Q, L, D, ν, k*; $\Rightarrow$ find h_m

(b) *Flow rate* $\Rightarrow$ for given: h_m, *L, D, ν, k*; $\Rightarrow$ find *Q*

(c) *Diameter* $\Rightarrow$ for given: h_m, *Q, L, ν, k*; $\Rightarrow$ find *D*

∴ *Examples for solving this kind of problems, as well as problems concerning energy losses (linear and local) in pipeline systems will be presented on the tutorial classes.*

8.3. Incompressible flow in noncircular ducts

Some of flow patterns in noncircular ducts are shown in *Fig. 8.12*.

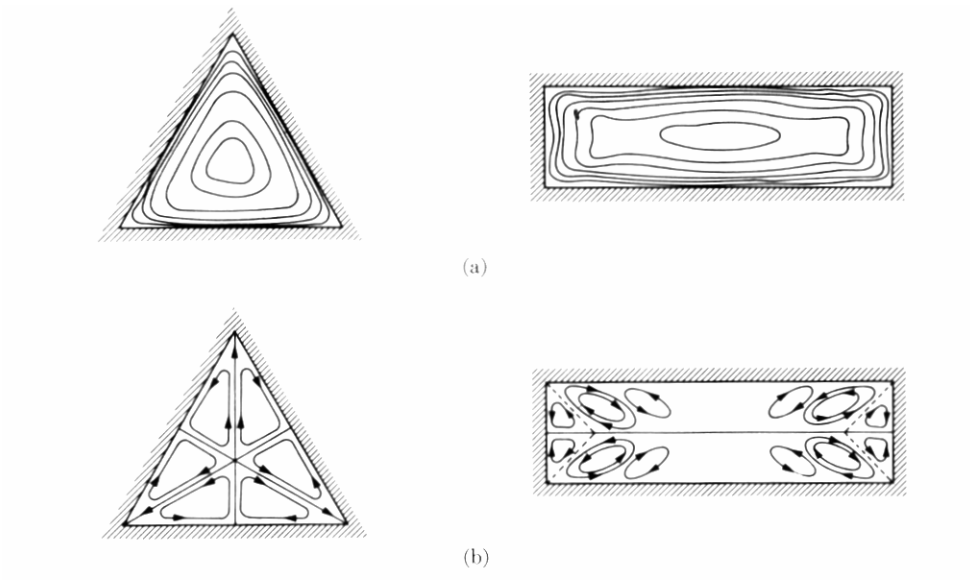


Fig. 8.12: Velocity contours and diagrams of secondary motions for fully developed flow in noncircular ducts: (a) velocity contours; (b) secondary circulation patterns [2].

Some theoretical and experimental investigations lead towards a conclusion that for the cases when the cross section has a ratio A/O close to circumscribing circle or semicircle, the head loss per unit lenght will be nearly the same as for a pipe. A/O = area/wetted perimeter.

This is the case for sections like squares, equilateral triangles, and ovals

∴ *The friction-loss data for circular pipes may be used.*

- **friction losses in closed conduits, two dimensional flows**

  - *Friction losses in clossed conduits*

The friction-loss data for circular pipes may be used.
The Darcy equation can be employed in a slightly different form.

Summing all forces in the flow direction, as on *Fig. 8.13*, ⇒

$$(p_1 - p_2)A = \tau_w OL$$

From the previously defined procedure - equations (8-20) to (8-24) ⇒ $h_m = \frac{\Delta p}{\gamma} = \tau_w \frac{Ol}{A\rho g}$

$$h_m = \lambda' \frac{L}{4R_h} \frac{v_{ave}^2}{2g} = \lambda' \frac{L}{D_h} \frac{V^2}{2g} \qquad (8\text{-}58)$$

Steady flow in a constant area conduit (as shown on Fig. 8.13) is cocidered.

∴ *The equation (8-24) can be used for linear head loss for flow in conduits with any shape cross-sections* - noncircular sections as well (see *Fig. 8.13* and *Fig. 8.5*).

$\lambda' = G\lambda$ - *corrected friction factor*

$\lambda' \approx \lambda$ - for turbulent flow;

$\lambda' = (0.4 \div 1.5)\lambda$ - for laminar flow

⇒ The previously explained procedure and diagrams (e.g., diagram on *Fig. 8.11*) for determining $\lambda = f = f(\mathrm{Re}, k_s / D)$ can be used as well, with taking into account that:

$$R_h = \frac{A}{O}; \qquad D = D_h = 4R_h = \frac{4A}{O}$$

$$\mathrm{Re} = \frac{4VR_h}{\nu} \quad \text{and} \quad \frac{k_s}{D} = \frac{k_s}{4R_h} \qquad (8\text{-}59)$$

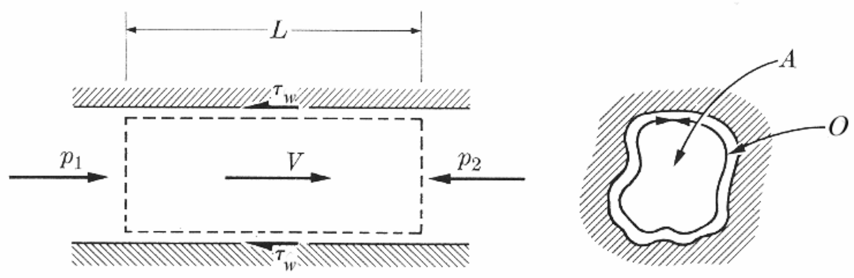


Fig. 8.13: Free-body diagram for steady flow in a constant-area conduit.

  - *Friction losses in 2-D flows;*

A flow between two plates (*Poiseuile flow*) can be treated as 2-D flow.
The basic equation for steady laminar flow are derived in chapter 6.3. ⇒ see equations (6-20) to (6-22).

The *friction factor for laminar flow between two plates* can be calculated from the expression:

$$\lambda = \frac{96}{\text{Re}} \tag{8-60}$$

$\text{Re} = \frac{4VR_h}{\nu}$; $\quad R_h = \frac{A}{O} = h$

For *2-D turbulent flow*, experimental results for *fully developed turbulent flows* in rectangular channels with cross-section as shown on *Fig. 8.12*, with $A:B = 60:1$ and $A:B = 12:1$, the friction laws can be expressed as follows:

for λ *in smooth channels*
$$\frac{1}{\sqrt{\lambda}} = 2.03\log\left(\frac{2BV}{\nu}\sqrt{\lambda}\right) - 0.47 \tag{8-61}$$

for λ *in rough channels*
$$\frac{1}{\sqrt{\lambda}} = 2.03\log\left(\frac{B/2}{k_s}\right) + 2.11 \tag{8-62}$$

$V = v_{ave}$; $\quad B = 2h$ -see *Fig. 8.14*.

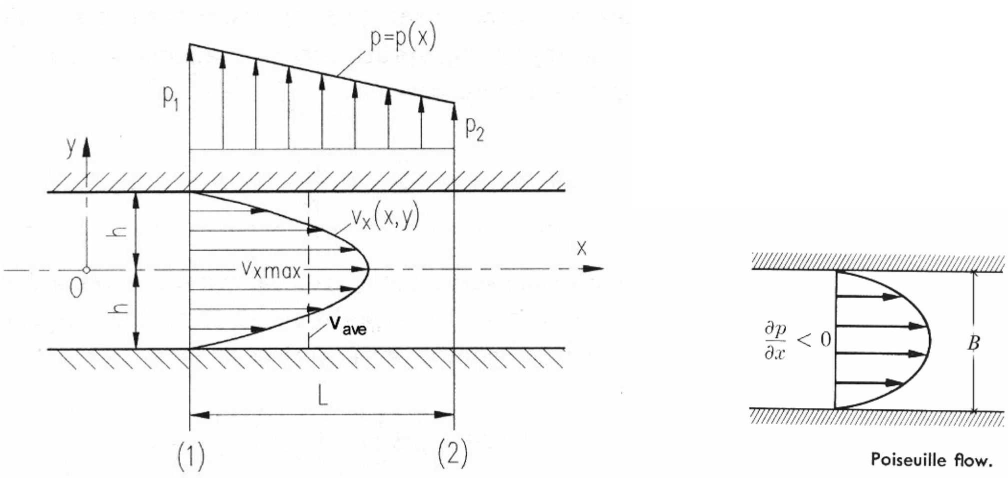


Fig. 8.14: Two dimensional flow between two plates

8.4. Flow in prismatic open channels

Open channel is a conduit in which a dense fluid flow under gravity with a definite interface separating it from an overlying lighter fluid.

Usually: dense fluid = liquid; overlying lighter fluid = gas ⇒ e.g., *water* and *air*.

⇒ *Free surface* = the interface between the liquid and the gas.

Natural open channels (e.g., *rivers* etc.) vary in size, shapes, and roughness ⇒ irregular nonuniform sections to the flow.

Artificial channels also vary in size, but have a narrower range of roughness ⇒ usually built with regular geometric shapes.
⇒ *Prismatic channels* = channels with constant channel section and bottom slope.
⇒ Rectangles, trapezoids, triangles, circles, parabolas and combinations are commonly used as prismatic channel sections.

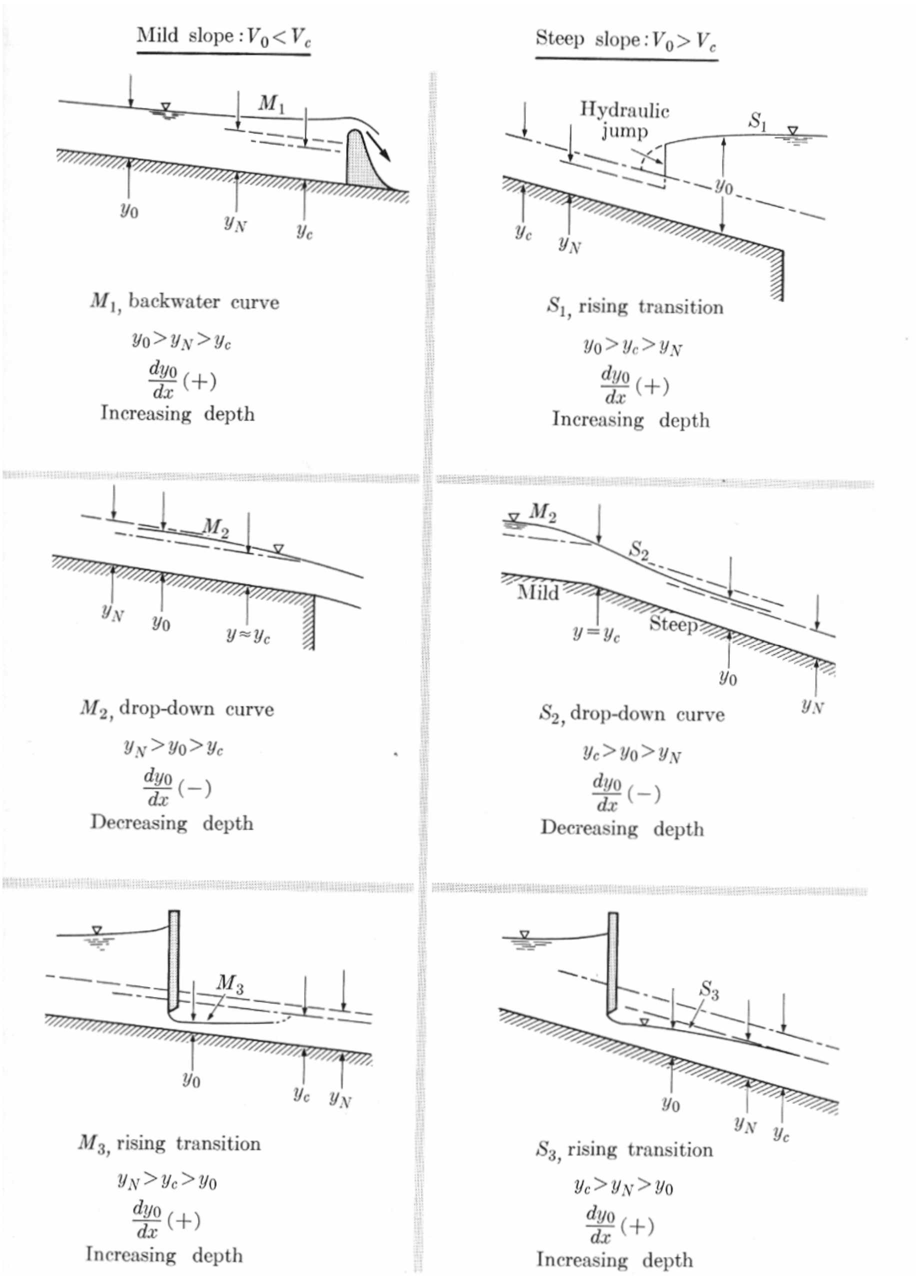


Fig. 8.15: Some open channel surface profiles

Open channels are, in general, noncircular.

Many open channels are wide ⇒ the velocity-friction relations can be examined on a two dimensional basis.

∴ *Usually, the relations among energy flux, momentum flux, flow depth, and friction are treated by one-dimensional analysis.*

∴ *A basic approach and basic equations, using one-dimensional relations for prismatic open channel, are presented in this chapter. .⇒ Uniform flow in a prismatic open channel is considered.*

- **one dimensional open-channel equations, head-loss equations**

The one-dimensional *total head* or *energy per unit weight H* (in Nm/N) for each fluid element ⇒

$$H = h + \frac{p}{\gamma} + \frac{V^2}{2g} \qquad (8\text{-}63)$$

$V = v_{ave}$; $\gamma = \rho g$;

The basic notation and expressions for flow in open channels are given in the table below, according Fig. 8.17.

A = cross-sectional area of channel

b = surface width = bottom width for rectangular channel

C = Chezy coefficient

$\frac{p}{\gamma} + h$ = piezometric head

H = total head $= \frac{p}{\gamma} + h + \frac{V^2}{2g}$

h = elevation above datum

h_0 = elevation of channel bottom

h_f = head loss due to surface resistance

H_L = total head loss

H_0 = specific head $= y_0 + \frac{V^2}{2g}$

L = length along slope ($dL = dx$)

n = roughness factor in Manning formula

P = wetted wall perimeter

$q = y_0 V$ = m³/sm

$Q = AV$ = m³/s

$R_h = \frac{A}{P}$ = hydraulic radius

$S = -\frac{d(y_0 + h_0)}{dx}$ = slope of free surface

$S_H = -\frac{dH}{dx}$ = slope of energy grade line

$S_0 = \sin\alpha_0$ = bottom slope $= -\frac{dh_0}{dx}$

V = average velocity corresponding to depth y_0

V_c = critical velocity corresponding to critical depth y_c

V_N = average velocity corresponding to normal depth y_N

x = distance in flow direction

y_0 = actual depth; y_c = critical depth; y_N = normal depth — for $\alpha_0 < 10°$, depth is taken as vertical distance which is satisfactory approximation

Conjugate depths = depths before and after a hydraulic jump

Alternate depths = subcritical and supercritical depths at the same specific head

$P = O$ - according previous notation ⇒ $R_h = A/O$

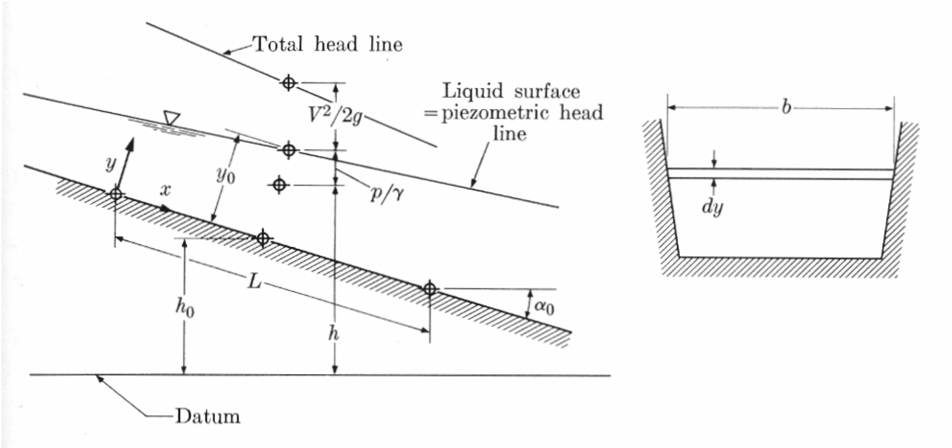


Fig. 8.17: Notations for one-dimensional open channels

According *Fig. 8.17*, and the given notations, it is assumed that:

- *The flow is uniform or gradually varying in the flow direction* $\Rightarrow$
The following quantities can be neglected: acceleration normal to the bottom, static pressure variation due to turbulence. $\Rightarrow$

$\frac{p}{\gamma} + h = const$, over a normal to the channel floor ("y" direction), $\Rightarrow$ according *Fig. 8.17*:

bottom slope $\qquad \sin\alpha_0 = -dh_0/dx = S_0$ (8-64)

Since, from Fig. 8.17 $\Rightarrow$ y_0 - depth; $(p/\gamma)_0 = y_0\cos\alpha_0$ - pressure head on the channel floor $\Rightarrow$ the total head equation (8-63) is transformed into:

$$H = h_0 + y_0\cos\alpha_0 + \frac{V^2}{2g} \tag{8-65}$$

For small slopes (e.g. $\alpha_0 < 10^0$, $S_0 < 0.018$) $\Rightarrow$ $\cos\alpha_0 \approx 1$ $\Rightarrow$

$$H = h_0 + y_0 + \frac{V^2}{2g} = h_0 + H_0 \tag{8-66}$$

$H_0 = y_0 + V^2/2g$ - *specific head.*

$\therefore$ *The head loss on a distance L will be:*

$$H_L = H_1 - H_2 = (h_0 + H_0)_1 - (h_0 + H_0)_2 = \int_0^L \frac{dH}{dx}dx \tag{8-67}$$

$dH/dx = -S_H$ - *energy grade line slope.*

Differentiating eq. (8-66) $\Rightarrow$ $\frac{dH}{dx} = \frac{dh_0}{dx} + \frac{dH_0}{dx}$ $\Rightarrow$ $-S_H = -S_0 + \frac{dH_0}{dx}$ $\Rightarrow$

$$\frac{dH_0}{dx} = \frac{dH_0}{dy_0}\frac{dy_0}{dx} = S_0 - S_H \tag{8-68}$$

Basic differential equation for one-dimensional open channel flow: $\frac{dy_0}{dx} = \frac{S_0 - S_H}{dH_0/dy_0}$ (8-69)

For steady *uniform flow* $\Rightarrow$ $y_0 = const$; $V = const$; $\Rightarrow$ $S = S_H = S_0$ $\Rightarrow$ $H_0 = const$

∴ *The head loss equation reduces* (from eq. (8-67)) to:

$$H_L = h_{01} - h_{02} = S_H L = S_0 L = h_f \tag{8-70}$$

Comparing the equation (8-70) with equation (8-27), it is obvious that:

$$I = S = \frac{h_f}{L} = \frac{h_m}{L} = \frac{(p_1 - p_2)/\gamma}{L} = \frac{h_1 - h_2}{L} = \sin\alpha_0$$

$I = S$ *- hydraulic gradient or slope*; $h_m = h_f$ *- linear head loss*.

Free surfaces are subjects to gravity waves $\Rightarrow$ *"c" - celerity = speed of the wave.*
The free surface behavior = f(V/c).

For *elementary gravity waves* (with depths small compared to wavelength) $\Rightarrow$

$$c = \sqrt{g y_0} \tag{8-71}$$

Froude number for open channels:

$$\mathrm{Fr} = \frac{V}{\sqrt{g y_0}} \tag{8-72}$$

$\mathrm{Fr} = 1 \Rightarrow V = c$ *- critical velocity*;

$\mathrm{Fr} < 1 \Rightarrow V < c$ *- subcritical*; $\mathrm{Fr} > 1 \Rightarrow V > c$ *- supercritical*

- **velocity and friction laws for two-dimensional channels, computation examples**

- *Head loss, friction factor and average velocity*

Darcy equation expressed in the form for noncircular conduits is widely used $\Rightarrow$ *the hydraulic radius and Chezy-Maning formulae are applied* - see equations (8-21) to (8-26) in chapter 8.1.
$\Rightarrow$

$$h_f = h_m = \lambda \frac{L}{4R_h} \frac{V^2}{2g} \tag{8-73}$$

$$R_h = \frac{A}{O} = \textit{flow cross section/wetted perimeter}$$

$$\lambda = \lambda(v, \rho, \mu, k, \text{channel size, chanel shape})$$

∴ For channel sections R_h close to that of a circumscribing circle or semicircle, λ can be evaluated from the pipe-friction diagram (*Fig. 8.11*).

$\Rightarrow$ To use *Fig. 8.11*, $\Rightarrow$

$$\mathrm{Re} = \frac{4VR_h}{\nu}; \quad \frac{k_s}{D} = \frac{k_s}{4R_h}$$

∴ For very wide channels, the pipe-friction factors become less applicable!

∴ Most open channels are physically large compared to pipes and other closed ducts. $\Rightarrow$
Re $\rightarrow$ *very large*, $\Rightarrow$ *turbulent flow in fully rough regime* $\Rightarrow$

$$\lambda = \lambda\left(\frac{k_s}{D_h}\right)$$

$\therefore$ *Chezy and Manning formulae* are widely applied - see equations (8-25) and (8-26) $\Rightarrow$

$$V = v_{ave} = \sqrt{\frac{8g}{\lambda}}\sqrt{R_h S} = C\sqrt{R_h S} \tag{8-74}$$

$S = S_0 = S_H = \dfrac{h_f}{L} - dH/dx$ - *hydraulic gradient or slope* (see equation (8-70));

$C = \sqrt{8g/\lambda} = f(v, \rho, \mu, k, \text{channel size, chanel shape})$ - *flow resistance factor*

Manning derived corresponding formulae:

$$C = \frac{1}{n}R_h^{1/6} = \frac{1}{n}\left(\frac{D_h}{4}\right)^{1/6} \tag{8-75}$$

$$\lambda = \frac{8g}{C^2} = \frac{8gn^2}{R_h^{1/3}} \tag{8-76}$$

$$V = \frac{1}{n}R_h^{2/3}S^{1/2} \tag{8-77}$$

n - Manning's roughness coefficient; depends of the conduit roughness - see the *Literature*.

$n = 0.040$ - for very rough surfaces (earth with weeds and stones);

$n = 0.012$ - for normally rough surfaces (finished concrete).

- *Velocity profile:*

Consider:
- open channel whose width is many times its depth;
- the flow is approximately two-dimensional;
- fully developed velocity profiles for steady uniform flow;
- Velocity profiles are logarithmic (as found for pipes) - see *Fig. 8.3A*(b), and *Fig. 8.18*. (see about *Prandtl* and *Karman* theories in chapter 8.2).

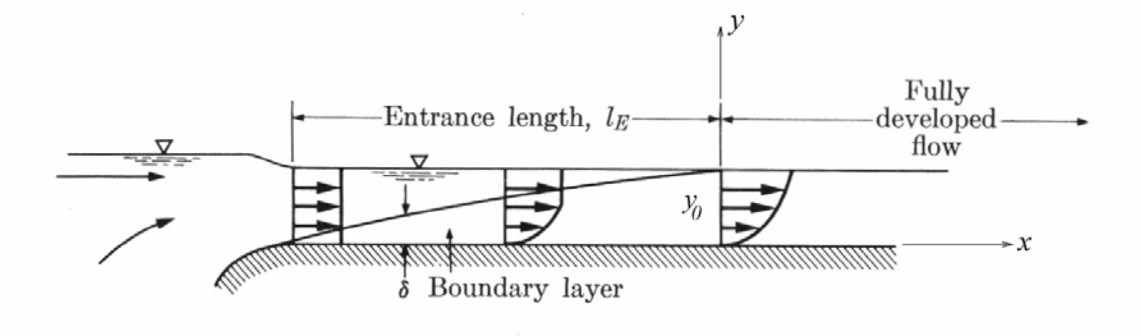


Fig. 8.18: Velocity profile for steady uniform 2-D flow in open channel

$$\frac{\bar{v}_{max} - \bar{v}_x}{v_\tau} = -\frac{2}{k}\log\frac{y}{y_0} \tag{8-78}$$

$v_\tau = \sqrt{\tau_w/\rho} = \sqrt{gR_hS_0}$; $k \approx 0.4$ - *Karman's constant*

- *Computation examples:*

The three basic problems for open channel flow are:

(a) *Channel slope (Head loss)*

Given: *Q, L,* $\nu = \mu / \rho$*, size, shape, and roughness.* $\Rightarrow$ Find: $S_0 = S_H$

(b) *Flow rate*

Given: *S_0, L,* $\nu = \mu / \rho$*, size, shape, and roughness.* $\Rightarrow$ Find: *V* and *Q*

(c) *Size (R_h for a given shape)*

Given: *S_0, Q, L,* $\nu = \mu / \rho$*, size, shape, and roughness.* $\Rightarrow$ Find: R_h

These problems are solved with steps analogous to pipe flow problems $\Rightarrow$ *examples will be presented on the tutorial classes*.

$\therefore$ *In all three problems, the fundamental step is determination of* λ.

$\Rightarrow$ Application of the *Darcy pipe friction approach* or *Chazy approach.*

8.5. Immersed bodies, drag and lift

The investigation of the drag and lift concepts are very important for various fields oh Fluid Mechanics application: aeronautics, turbo machinery, multicomponents flows, chemical reactions etc.

- *hydrodynamic forces and force coefficients, drag of symmetrical bodies, lift and drag of nonsymmetrical bodies*

Some of the definition from chapter 6.5 are repeated here:

Drag (sometimes called resistance) is the force that resists the movement of a solid object through a fluid in the direction of its movement - in this case the object is moving in a quiescent fluid.

Drag force (F_D) can be also defined as the acting force of the fluid flow on a immersed body, in the direction of the flow relative velocity V_0 - see *Fig. 6.12*.

$\therefore$ *The total drag force F_D* is defined as (see *Fig.6.12*):

$$F_D = D = C_D \rho \frac{V_0^2}{2} A \tag{8-79}$$

A - frontal area normal to $V_0 \Rightarrow$ $A = A_P$

$\therefore$ *The total lift* is defined as (see *Fig.6.12*):

$$F_L = L = C_L \rho \frac{V_0^2}{2} A \tag{8-80}$$

C_L - lift coefficient;
A - the planform area of a wing (largest projected area of the body, or the projected area normal to V_0.

$\therefore$ *C_D and C_L are usually experimentally obtained.*

$\therefore$ *CFD application in solving numerical models with the contemporary PCs, using the experimental verifications, give reasonable results.*

Following the dynamic similitude (see chapter 7) ⇒

$$C_D = C_D(\text{geometry}, \text{Re}, \text{Fr}, \text{M}) \tag{8-81}$$

$$C_L = C_L(\text{geometry}, \text{Re}, \text{Fr}, \text{M}) \tag{8-82}$$

For example, consider the drag coefficient for characteristic flow and fluid conditions ⇒:

- *Incompressible fluids in enclosed systems:* $C_D = C_D(\text{geometry}, \text{Re})$
- *Incompressible fluids in systems having an interface:* $C_D = C_D(\text{geometry}, \text{Re}, \text{Fr})$
- *Compressible fluids:* $C_D = C_D(\text{geometry}, \text{M})$

⇒ Some data for the *drag coefficients for symetric bodies* are shown in the diagrams on *Fig. 8.19*. See *Literature*!

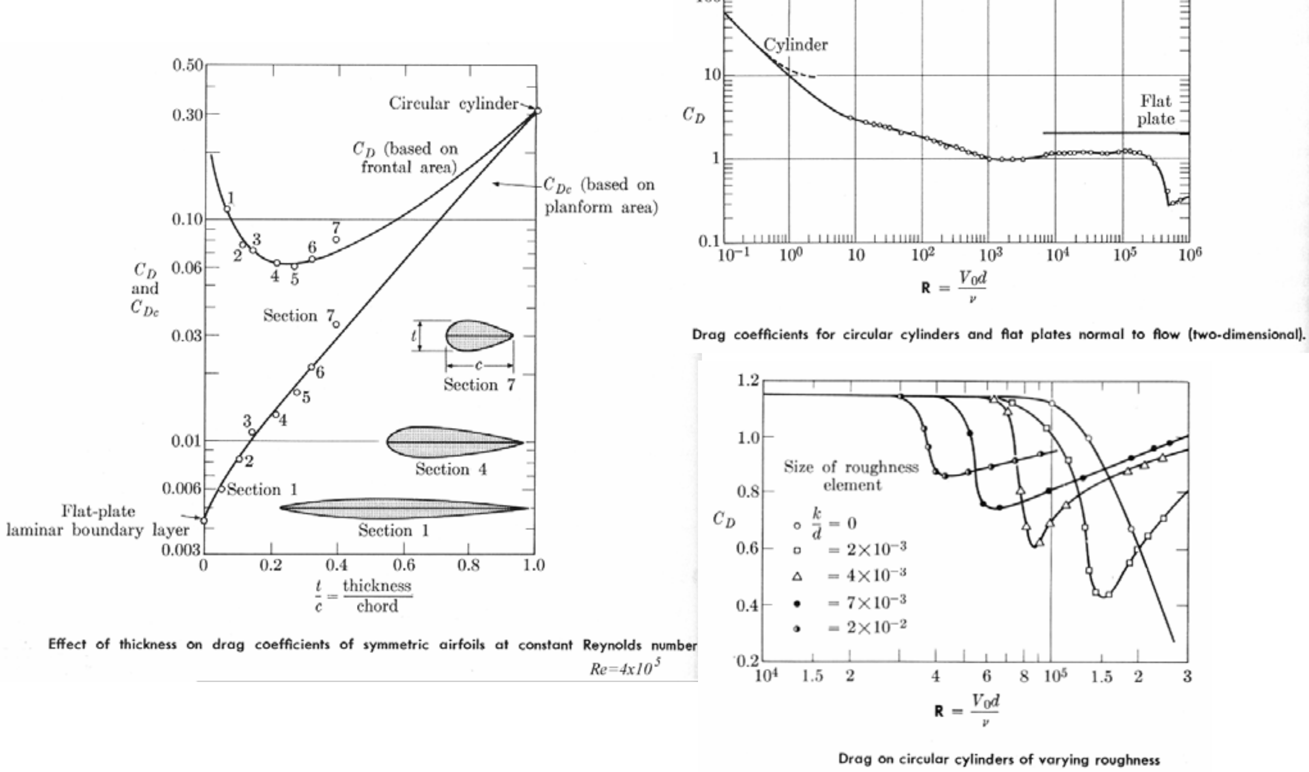
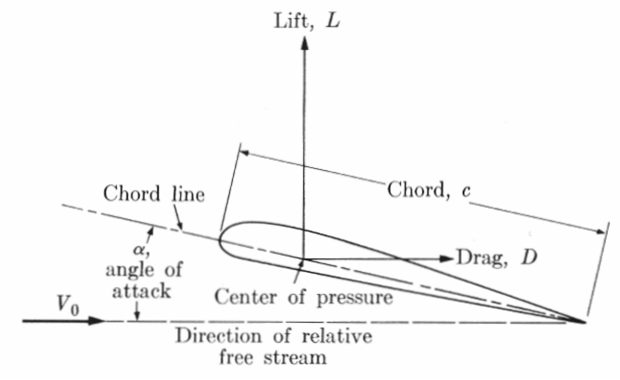


Fig. 8.19: Some data for the drag coefficients for symetric bodies

Some experimental data for the *lift and drag coefficients for nonsymetric body* are shown in the diagrams on *Fig. 8.20*. ⇒ See *Literature*!

Recently, numerous data of numerical CFD solving of the governing equations are available!

⇒



Airfoil geometry

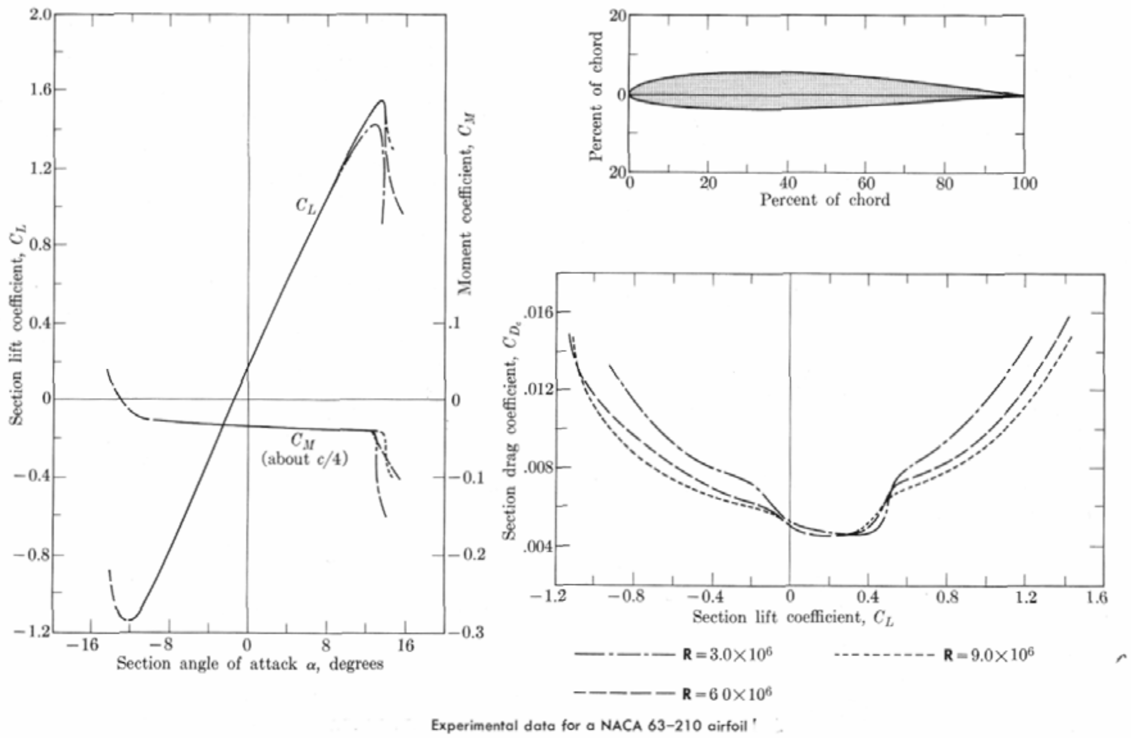


Fig. 8.20: Experimental data for the lift and drag coefficients for an airfoil

8.6. Basic approach to turbulent jets and diffusion processes

- *free turbulence, diffusion processes in nonhomogeneous fluids*

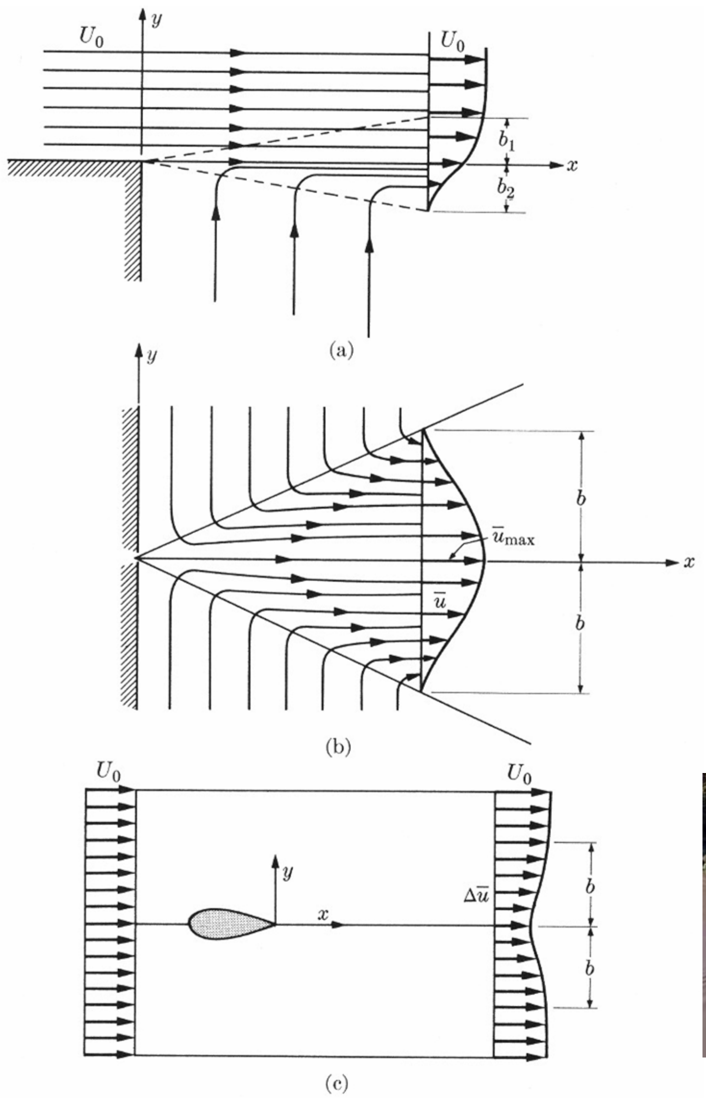
The term *wall turbulence* is used to describe turbulence generated in velocity gradients caused by the no-slip condition.

The term free turbulence, on the other hand, describes turbulent motions which are not affected by the presence of solid boundaries.

Some examples of free turbulent flows are shown in *Fig. 8.21 and Fig. 8.22:*

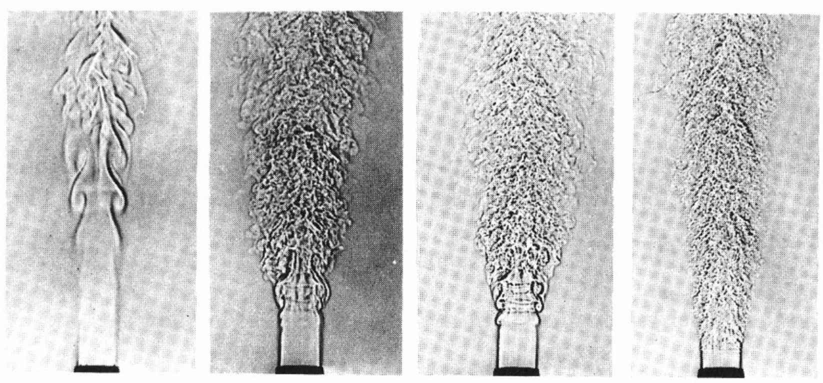
(a) the spreading of the edge of a plane jet;
(b) (b) a round jet issuing from a slot into a surrounding fluid of the same phase (water into water or air into air); and
(c) (c) the flow in the wake of an immersed body.

In all cases velocity gradients are generated. If the Reynolds numbers are sufficiently high, the flow is unstable and zones of turbulent mixing are developed.



Free turbulent flows: (a) spreading at the edge of a plane jet; (b) velocity distributio developed immersed jet; (c) velocity distribution in the wake of an immersed object.

Fig.8.21: Free turbulent flows



Development of a turbulent jet

Fig. 8.22: Development of a turbulent jet

Diffusion is the spontaneous net movement of particles from an area of high concentration to an area of low concentration in a given volume of fluid (either liquid or gas) down the concentration gradient.

For example, diffusing molecules will move randomly between areas of high and low concentration but because there are more molecules in the high concentration region, more molecules will leave the high concentration region than the low concentration one.
Therefore, there will be a net movement of molecules from high to low concentration. Initially, a concentration gradient leaves a smooth decrease in concentration from high to low which will form between the two regions. As time progresses, the gradient will grow increasingly shallow until the concentrations are equalized.

∴ *Diffusion is a characteristic process for turbulent jets, and turbulent buoyant jets and plumes!*

∴ *In hydrodynamics, a plume is a column of one fluid moving through another*- see *Fig.8.23 and Fig. 8.24.*

∴ *A thermal plume is one which is generated by gas rising from above heat source. The gas rises because thermal expansion makes warm gas less dense than the surrounding cooler gas.*

⇒ *Some flow field characteristics of buoyant jets and plumees can be seen on Figres 8.23.*

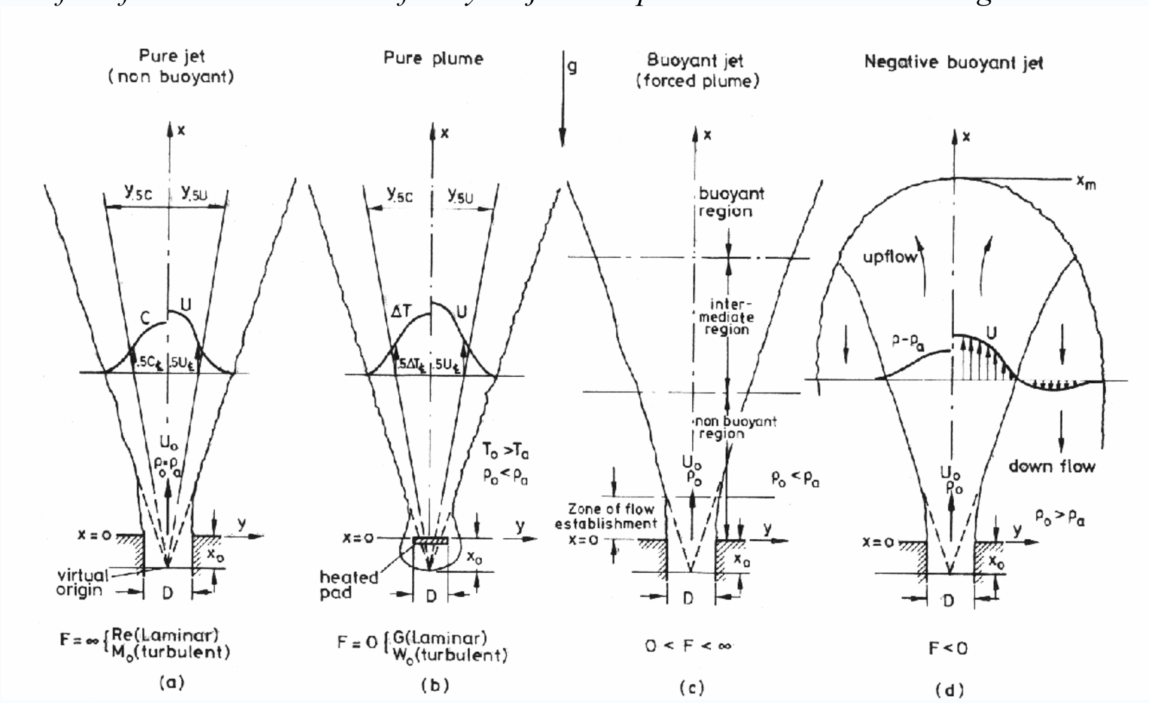


Fig. 8.23 : Buoyant jets in uniform surrounding

⇒ Several effects control the motion of the fluid, including:
momentum, *buoyancy* and *density difference*.

- *When momentum effects are more important than density differences and buoyancy effects, the plume is usually described as a jet - buoyant jet.*

- Usually, as a plume moves away from its source, it widens because of *entrainment of the surrounding fluid at its edges.*
This usually causes a plume which has initially been 'momentum-dominated' to become 'buoyancy-dominated' (this transition is usually predicted by a dimensionless number called the Richardson number).

- A further phenomenon of importance is whether a plume is in *laminar flow* or *turbulent flow*. Usually there is a *transition from laminar to turbulent* as the plume moves away from its source.

This phenomenon can be clearly seen in the rising column of smoke from a cigarette.

- Another phenomenon which can also be seen clearly in the flow of smoke from a cigarette is that the leading-edge of the flow, or the starting-plume, is quite often approximately in the shape of a ring-vortex (smoke ring).

∴ *Plumes and buoyant jets are of considerable importance in the dispersion of air pollution -* see *Fig. 8.24.*

The problem of reducing the pollution of our water bodies and of the atmosphere has been and still is a serious problem; and concerns to legislators, scientists and engineers.

In order to minimize the impact of some unavoidable emission of pollutants into our environment, *the dispersion of pollutants should be predictable*.

The fluid motion governing this dispersion is mostly turbulent and under gravitational influence, it is important to study turbulent buoyant flows and to develop reli able methods for their prediction.

A number of methods have been proposed for calculating the practically important cases of turbulent buoyant jets and plumes, ranging from simple empirical formulae to *complex models involving partial differential equations* - see chapter 6.

Experimental data are required by all the methods, either as a direct basis for the empirical formulae or to determina empirical constants or functions appearing in the methods. They are also needed to define the range of validity of a method.

Simple Plume Modelling

Quite simple modelling will enable many properties of fully-developed, turbulent plumes to be investigated.

1) It is usually sufficient to assume that the pressure gradient is set by the gradient far from the plume (this approximation is similar to the usual Boussinesq approximation)

2) The distribution of density and velocity across the plume are modelled either with simple Gaussian distributions or else are taken as uniform across the plume (the so-called 'top hat' model).

3) Mass entrainment velocity into the plume is given by a simple constant times the local velocity - this constant typically has a value of about 0.08 for vertical jets and 0.12 for vertical, buoyant plumes. For bent-over plumes, the entrainment coefficient is about 0.6.

4) Conservation equations for mass flux (including entrainment) and momentum flux (allowing for buoyancy) then give sufficient information for many purposes.

For a simple rising plume these equations predict that the plume will widen at a constant half-angle of about 6 to 15 degrees.

A top-hat model of a circular plume entraining in a fluid of the same density ρ is as follows:

The Momentum M of the flow is conserved so that:

$$A\rho v^2 = M = const$$

The mass flux J varies, due to entrainment at the edge of the plume, as

$$dJ / dx = dA\rho v / dx = kr\rho v$$

where k is an entrainment constant, r is the radius of the plume at distance x, and A is its cross-sectional area.

∴ This shows that the *mean velocity v* falls inversely as the radius rises, and the plume grows at a constant angle $dr/dx = k'$.

Atmospheric dispersion modeling

Atmospheric dispersion modeling is the mathematical simulation of how air pollutants disperse in the ambient atmosphere.
It is performed with computer programs that solve the mathematical equations and algorithms which simulate the *pollutant dispersion*.
The dispersion models are used to estimate or *to predict the downwind concentration of air pollutants emitted from sources such as industrial plants and vehicular traffic*.

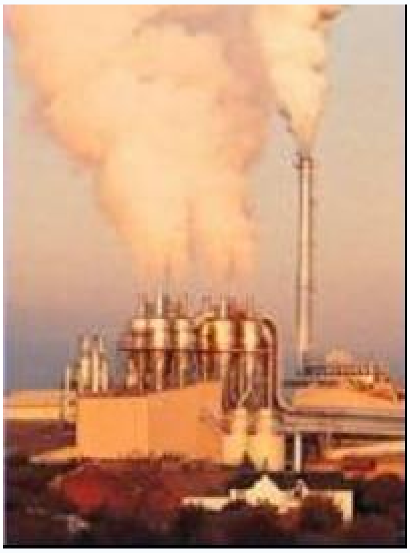
Such models are important to governmental agencies tasked with protecting and managing the ambient air quality.
The models are typically employed to determine whether existing or proposed new industrial facilities are or will be in compliance with the National Ambient Air Quality Standards (NAAQS) in the United States and other nations.
The models also serve to assist in the design of effective control strategies to reduce emissions of harmful air pollutants.

∴ Both theoretical and experimental methods have been widely applied for buoyant jets and plumes flows quantities determination.

∴ *The CFD approach for solving the governing equations of different flow casesinduced by buoyan jets and plumes is widely used* ⇒ see chapter 6.7.

The use of sophisticated PC (even so-called "*super computers*") and software packages enable solving of numerical models, for which extremely long execution time was needed in the past (or it was impossible to be solved). Reducing many of the aproximations that were needed.

⇒ *Some results of CFD solving the buoyant jets governing equations are given on Fig. 8.25:*



Industrial air pollution plumes

Large Natural Convection Plume

Fig. 8.24 : Some examples of turbulent buoyant jets

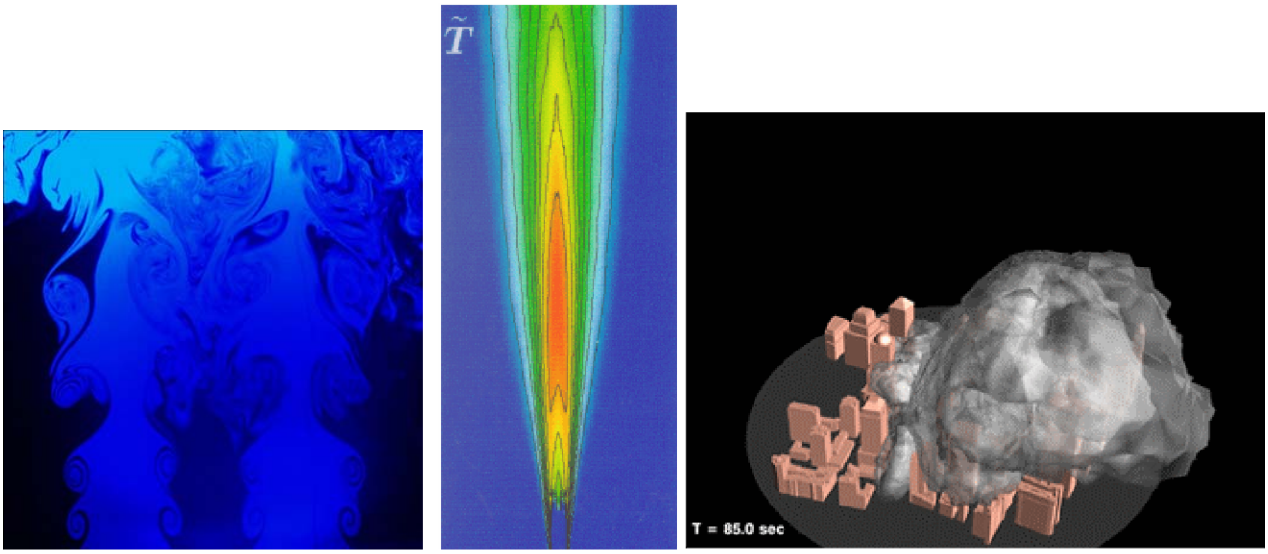


Fig. 8.25: Some results of CFD solving the buoyan jets governing equations

8.7. Basic approach to multiphase flow

In fluid mechanics, *multiphase flow* is a generalisation of the modelling used in two-phase flow to cases where the two phases are not chemically related (e.g. dusty gases) or where more than two phases are present (e.g. in modelling of propagating steam explosions).

Each of the phases is considered to have a separately defined volume fraction (the sum of which is unity), and velocity field. Conservation equations for the flow of each species (perhaps with terms for interchange between the phases), can then be written down straightforwardly.

The momentum equation for each phase is less straightforward.
It can be shown that a common pressure field can be defined, and that each phase is subject to the gradient of this field, weighted by its volume fraction.
Transfer of momentum between the phases is sometimes less straightforward to determine, and in addition, a very light phase in bubble form has a virtual mass associated with its acceleration. (The virtual mass of a single bubble is about half its displaced mass).

These terms, often called constitutive relations, are often strongly dependent on flow regime.

Two-phase flow is a particular example of multiphase flow.

In fluid mechanics, two-phase flow occurs in a system containing gas and liquid with a meniscus separating the two phases.

Historically, probably the most commonly-studied cases of two-phase flow are in large-scale power systems. Coal and gas-fired power stations used very large boilers to produce steam for use in turbines.
In such cases, pressurised water is passed through heated pipes and it changes to steam as it moves through the pipe.
The design of boilers requires a detailed understanding of two-phase flow heat-transfer and pressure drop behaviour, which is significantly different from the single-phase case.
Even more critically, nuclear reactors use water to remove heat from the reactor core using two-phase flow.

A great deal of study has been performed on the nature of two-phase flow in such cases, so that engineers can design against possible failures in pipework, loss of pressure, and so on (a loss-of-coolant accident (LOCA)).

Another case where two-phase flow can occur is in pump cavitation.
Here a pump is operating close the vapor pressure of the fluid being pumped. If pressure drops further, which can happen locally near the vanes for the pump, for example, then a phase change can occur and gas will be present in the pump. Similar effects can also occur on marine propellors; wherever it occurs, it is a serious problem for designers. When the vapor bubble collapses, it can produce very large pressure spikes, which over time will cause damage on the propellor or turbine.

The above two-phase flow cases are for a single fluid occurring by itself as two different phases, such as steam and water. The term 'two-phase flow' is also applied to mixtures of different fluids having different phases, such as air and water, or oil and natural gas. Sometimes even *three*-phase flow is considered, such as in oil and gas pipelines where there might be a significant fraction of solids.

Other interesting areas where two-phase flow is studied includes in climate systems such as clouds, and in groundwater flow, in which the movement of water and air through the soil is studied.

Other examples of two-phase flow include bubbles, rain, waves on the sea, foam, fountains, mousse, and oil slicks.

Several features make two-phase flow an interesting and challenging branch of fluid mechanics:

- Surface tension makes all dynamical problems nonlinear (see Weber number).
- In the case of air and water at Standard Temperature and Pressure, the density of the two phases differs by a factor of about 1000. Similar differences are typical of water liquid/water vapor densities.
- The sound speed changes dramatically for materials undergoing phase change, and can be orders of magnitude different. This introduces compressible effects into the problem.
- The phase changes are not instantaneous, and the liquid vapor system will not necessarily be in phase equilibrium.

∴ Both theoretical and experimental methods have been widely applied for multiphase flows quantities determination.

∴ *The CFD approach for solving the governing equations of different multiphase flows cases is widely used* ⇒ see chapter 6.7.

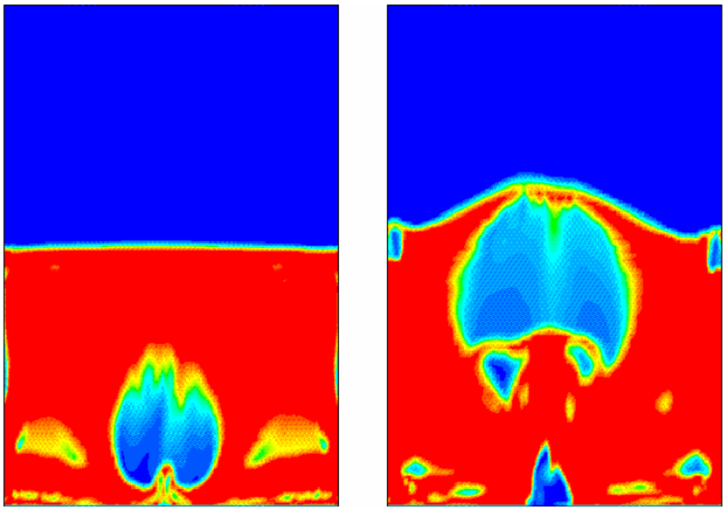
CFD has been used to improve process design by allowing engineers to simulate the performance of alternative configurations, eliminating guesswork that would normally be used to establish equipment geometry and process conditions.

A CFD analysis yields values for pressure, fluid velocity, temperature, and species or phase concentration on a computational grid throughout the solution domain.

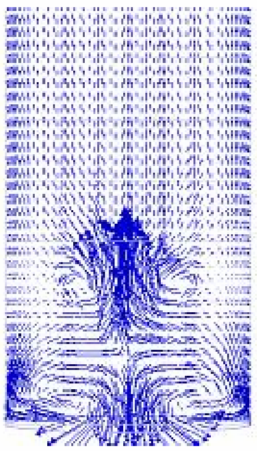
Example:

Major advancements in the area of *gas-solid multiphase flow modeling* offer substantial process improvements that have the potential to significantly improve *process plant operations*.

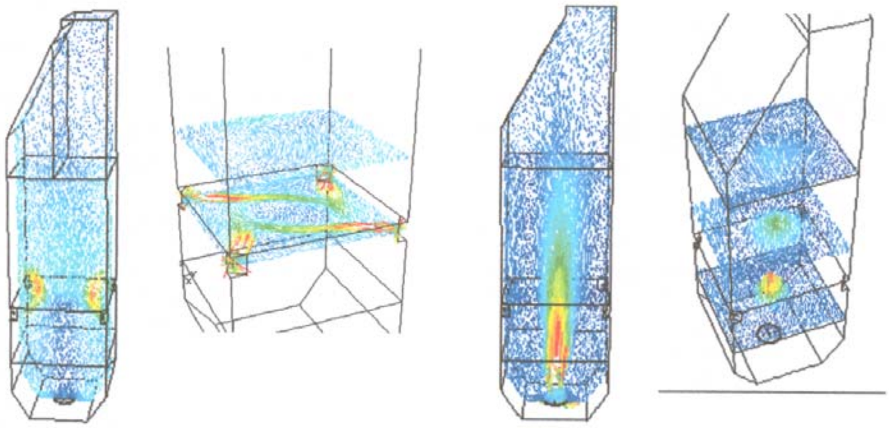
Prediction of gas-solid flow fields, in processes such as pneumatic transport lines, risers, fluidized bed reactors, hoppers and precipitators are crucial to the operation of most process plants.



An initially stationary bed of solids is fluidized by the action of a central jet.
Red indicates regionsof maximum solids volume fraction(~0.6), and blue indicates regions of maximum air volume fraction (1.0).



Flow field of a process in a fluidised bed



Velocity vectors in a boiler furnace

Fig 8.26: Some results of Fluent CFD solving of fundamental equations of some multiphase flow processes

COURSE LEARNING MATERIALS

Textbook

Street R.L., Watters G.Z., Vennard J.K., *Elementary Fluid Mechanics*, John Wiley & Sons, 7th editiond, 1996, ISBN: 978-0-471-01310-5

Bundalevski T., : *Mechanics of Fluids* (in Macedonian), University Ss Cyril and Methodius, publisher MB-3, Skopje, 1995, ISBN 9989-704-01-5

Nošpal A., Stojkovski V.,: *Fluid Mechanics - Prepared lecures and tutorials material for the DEREC subject*, edition of the Faculty of Mechanical Engineering, Skopje, 2007/2008

Professors from EU DEREC Universities: *Fluid Mechanics - Prepared lecures and tutorials material for the DEREC subject*, Educational Material prepared by professors from EU DEREC Universities, 2007/2008

Tutorial

Nošpal A., Stojkovski V.,: *Fluid Mechanics - Prepared lecures and tutorials material for the DEREC subject*, edition of the Faculty of Mechanical Engineering, Skopje, 2007/2008

Professors from EU DEREC Universities: *Fluid Mechanics - Prepared lecures and tutorials material for the DEREC subject*, Educational Material prepared by professors from EU DEREC Universities, 2007/2008

Lab practicum

Nospal A.: "*Fluid Flow Measurments and Instrumentation*" (in Macedonian), University Ss Cyril and Methodius, publisher MB-3, Skopje, 1995, iSBN 9989-704-02-3

Stojkovski V., Nošpal A., Kostic.Z.,: "*Practicum for Laboratory Works for the Subject Fluid Flow Measurements and Instrumentation*", edition for students of the Faculty of Mechanical Engineering, Skopje, 1993.

Stojkovski V., Nošpal A.,: *Fluid Mechanics - Prepared lecures and tutorials material for the DEREC subject*, edition of the Faculty of Mechanical Engineering, Skopje, 2007/2008

Professors from EU DEREC Universities: *Fluid Mechanics - Prepared lecures and tutorials material for the DEREC subject*, Educational Material prepared by professors from EU DEREC Universities, 2007/2008

Web support

http://www.derec.ukim.edu.mk

BACKGROUND

TEMPUS JEP DEREC MATERIALS

UNIVERSITIES CONSORTIUM: *University of Florence, University Sts. Cyril and Methodius, Aristotele University of Thessaloniki, Ruhr University Bochum, Vienna University of Technology*

Fluid Mechanics

Theory Homework No. 1

1. What are the definitions for *fluid* and *fluid mechanics*?
2. Define the dimensional homogeneity!
3. Which are the *basic definitions* and *formulae* for *pressure*, *temperature*, *density*, *specific weight* and *viscosity*?
4. Which are the *dimensional formulae* and *SI units* for *pressure*, *temperature*, *density*, *specific weight* and *viscosity*?
5. Write down the definitions, corresponding formulae and SI units for *specific heat*, *specific internal energy*, *and specific enthalpy*!
6. Define *compressibility*, *bulk modulus of elasticity* and *velocity of sound*! Write down the corresponding formulae!
7. Give the basic definitions for *vapor pressure* (*cavitation pressure*) and *surface tension*!
8. Give the definition for an *equation of state*! Which are the basic equations of state for *liquids* and *gasses*?
9. What kinds of forces generally act on a fluid element?
10. Define and write down the basic expressions for *body force* and *surface force*!
11. Give the definition and basic equatioin for *pressure*!
12. Define the *hydrostatic pressure*! Which are the *two important characteristics* ohf hydrostatic pressure?
13. Write down the expressions for *elementary pressure force* and *resultant pressure force*!
14. Derive the *Euler's equation* - *Fig. 2-3*!
15. Define the *potential of a force* and *equipotential surface*!
16. Derive the basic equations for *equilibrium in gravity field*!
17. Derive basic equations for *equilibrium of incompressible fluid in gravity field* - *Fig.2.6*! Define (write down the corresponding expressions) the *absolute pressure*, *gauge pressure* and *vacuum*.
18. For *open interconnected vessels* $p_1 = p_2 = p_a$ (*Fig. 2.7*), prove that $h_2 - h_1 = 0$!
19. For the hydrostatic manometers on *Fig. 2.8*, write down the expressions for *gauge pressure* and *vacuum*!
20. Write down the definition of the *Pascal's Law*, and prove it - *Fig 2.10*!
21. Derive the expression for the force on the piston K_1 ($P_1 = ?$) - *Fig. 2.11*, if the force on K_2 is $P_2 = \frac{a}{b} P$!
22. Derive the basic equations for *a container with linear movement with constant acceleration* - *Fig. 2.12*.
23. Derive the basic equations for *rotation of a liquid container around vertical axis* - *Fig. 2.13*.
24. What are the expressions for the *pressure force P*, and the coordinates of its *acting point D*? - *Fig. 2.14*.

25. For a case as on *Fig. 2.16*, write down thw exprwssions for the Pressure force P, and its componennt P_H and P_V!
26. Give the expressions for *pressure* and *pressure force* for each of the cases presented on *Fig. 2.17*!
27. What are the pressure force componenents acting on a curved surface - *Fig. 2.18, Fig. 2.19. Fig. 2.21*?
28. Give a definition for *buoyant force*! Which are the expressions for the acting forces on a immersed body - Fig. 2.23? Which one is the *Archimed force*?
29. Describe the cases shown on *Fig. 2.24* and *Fig. 2.25*!

Fluid Mechanics

Theory Homework No. 2

1. Define *velocity field*!
2. Why *Eulerian approach* has advantage compared to *Lagrangian*?
3. What is a *steady flow*?
4. Write down the expressions for velocity components in *Cartesian* and *polar* coordinate system!
5. Define *streamline* and *pathline*? What is the difference between them in case of steady flow?
6. Express the velocity component using a stream funcion! Which is the stream funvtion along a streamline?
7. Define *stream tube*!
8. Write down the *rate of change* of the velocity *in the x-direction* (*total derivative*)! Indicate the *velocity gradients* and "*local" change* in the expression.
9. Write down the expressions for *volume flow rate* and *mass flow rate*!
10. Derive the continuity equation (*Fig. 3.16*)!
11. Write down the *equations of continuity* for *unsteady compressible fluid flow* and *steady incompressible fluid flow*.
12. Write down the expressions for the *acceleration vector* and its *components* for *3-D fluid flow*.
13. Write down the expressions for *total velocity derivative* and *acceleration* for *one dimensional flow along* a stream line "*s*" - *Fig. 3.17*!
14. Define *relative*, *periferial* and *absolute velocity* for a *flow along a rotating stream line* (*Fig. 3.20*)! Write down the corresponding equations.
15. Write down the expression for the *overall acceleration vector* in case of *2-D flow with rotation axis normal to the flow plane* (*Fig. 3.20*)!

Fluid Mechanics

Theory Homework No. 3

1. Which are the acting forces in case of an inviscid fluid flow? Write down the basic expressions and basic vector equation.
2. Write down the *Bernoulli equation* for unsteady inviscid compressible fluid flow along a streamline (*Fig. 4.1a*).
3. Write down the *Bernoulli equation* for steady inviscid incompressible fluid flow. Explain its meaning according *Fig. 4.2*.
4. What is the pressure change if the stream line (*Fig. 4.1*) is a straight line ($r_k = \infty$), for steady inviscid incompressible fluid flow.
5. Write down the Bernoulli equation for compressible fluid flow along a rotating streamline (*Fig 4.4*). What is the difference if the fluid is incompressible?
6. Define irrotational (potential) fluid flow!
7. What are the expressions for velocity components v_x and v_y obtained with the potential and stream functions (*Fig. 4.5*)?
8. Write down the continuity equation in integral form for flow without singularities for compressible and incompressible fluid flow.
9. Write down the expressions for *Momentum Law* and *Moment of Momentum Law* for a closed control surface *K* bounding a mass *m* (see *Fig. 4.18*).
10. Give the definition of the *first law of thermodynamics*! Write down the corresponding equation that describes it.
11. What is the expression for *specific enthalpy*?

Fluid Mechanics

Theory Homework No. 4

1. Write down and explain the *continuity equation in integral form* and *Bernoulli's equation* for a *flow in a stream tube* (*Fig. 5.1* and *Fig. 5.2*).
2. What is the expression for the *Momentum Law for flow through stream tube* (*Fig 5.3*)? Give the expression for the r*esultant force* $\vec{F}_R$ acting on the fluid mass bounded by the control surface!
3. Give the expression for the acting force from the fluid to the solid boundaries - *Fig. 5.4*.
4. Derive the expression for volume flow rate (discharge) through a *Ventury tube* (*Fig. 5.5*).
5. Derive the *Torricelli's formula - Fig. 5.7*.
6. What is the general expression fror the entire discharge Q, for discharge into the atmosphere through large openings - *Fig. 5.10*.
7. What is the expression for the discharge through the entire opening in case of submerged discharge as on *Fig. 5.12.*
8. Write down the Bernoulli's equation from cross-section "*0*" to cross-section "*A*", and for the the rotating pipe (from "*A*" to "*2*") - *Fig. 5.16*.
9. Give the definition for cavitation!
10. Write down the *Bernoulli's equation for steady adiabatic fluid flow*! What is κ ?
11. What are the expressions for the force on bended pipe $\vec{F}_r$ - *Fig. 5.24*?
12. Write down the expression for the the reaction to the jet force F_{rx} - *Fig. 5.25*.
13. What is the expression for the *missile reaction force* - *Fig. 5.26*? Explain the procedure of obtaining the expression for the *missile velocity*!

Fluid Mechanics

Theory Homework No. 5

1. Define fluid shear stress, dynamic and kinematic viscosity.
2. Explain *Fig. 6.1*.
3. Give basic definitions for *laminar* and *turbulent flow*. Define *Reynolds number*.
4. Which is the procedure for obtaining the *Navier-Stockes equations* (give a general explanation)?
5. Which are the governing equations of viscous fluid laminar flow? For which cases the system of governing equations can be solved?
6. What are the approximations for solving the governing equations for the cases presented on *Fig 6.4* and *Fig. 6.5*? Which properties can be obtained?
7. Give the basic definition for *creeping fluid flow*! Write down the expression for the Drag force for the flow as on *Fig. 6.9* - name the properties in the expression!
8. Give the basic definition and characteristics concerning *boundary layer*!
9. Explain *Fig. 6.10*!
10. Give the definitions for *Drag force* and *Lift force - Fig. 6.12* and *Fig. 6.13*! Write down the general expressions for Drag and Lift force (name the properties in the expressions).
11. Give the definition, and write down the general expression for *Reynolds number* - name the properties in the expression! Define the *critical Reynolds number*!
12. What does the *mathematical model* (6-48) to (6-52) present? Write down the general expressions for the *instantaneous flow properties* u, v, w, *and* p!
13. Which are the *basic features of the theoretical method* for solving engineering problems? Give a short comment!
14. Which are the *basic features of the experimental method* for solving engineering problems? Give a short comment!
15. Shortly explain the *CFD approach*!

Fluid Mechanics

Theory Homework No. 6

1. Write down the *dimensional formulae* and SI *units* for: *acceleration, volume flow rate, circulation, kinematic viscosity, pressure, density, work, dynamic viscosity, bulk modulus of elasticity, mass flow rate, surface tension, quantity of heat, specific enthalpy*.
2. Derive the expression for volume flow rate in *Venturi meter* using the *Rayleigh's method*!
3. Show the *significance of the dimensionless groups* with the example of the use of Rayleigh's method for Ventiri meter.
4. Derive the expression for *flow in Venturi meter* using the *Vaschy's theorem*.
5. Write down the *fundamental scales for geometric, kinematic and dynamic similarity*. What is the meaning of the properties in the corresponding scale expressions?
6. Write down the expressions of similarity scales for flow gate, force and work. What is the meaning of the properties in the corresponding scale expressions?
7. Derive the *similarity criteria* for flow dominated by *viscous forces*!
8. Which are the *similarity criteria* for model and prototype in the s*ame gravity field* and with *same fluids*?

Fluid Mechanics

Theory Homework No. 7

1. How is treated a flow of liquids through pipes? What are the causes of viscous friction existence in this case?
2. Define the term *velocity profile*! Derive the expression for *average velocity* (*fig. 8.2*)!
3. Write down the basic equations for incompressible fluid flow in pipes!
4. Write down the *Darcy's formula*. What every member in the formula presents?
5. Derive the head loss expression according *Fig. 8.4*! Define the term of *hydraulic radius* and write down the corresponding formula!
6. Write down the equation for *linear head loss* for incompressible fluid flow in c*onduits with any shape cross-section*! What every member in the formula presents?
7. Write down the *Chezy formula for the average velocity* over a flow section! What every member in the formula presents?
8. Write down the expression for *hydraulic gradient* for *open channel flow* as on *Fig. 8.6*.
9. Write down the expression for *local head loss*! What every member in the formula presents? What is the general dependence of the *local head loss coefficient*?
10. Write down the expression for *total head loss* in a pipe line as shown on *Fig. 8.7*.
11. What expression is used for *pipe friction factor* λ in case of laminar flow? What is the magnitude of the *average velocity* in this case?
12. Explain the *Fig. 8.10*! For most common case what is approximately the magnitude of the *average mean velocity* $\bar{v}_{ave}$?
13. Write down the dependence formulae (general forms) for *friction factor of turbulent flow* in pipes - *smooth pipes*, *fully rough* and *transition zone!*
14. Write down the formula for *total head in one-dimensional open channel* (*Fig. 8.17*)! What every member in the formula presents?
15. What is the expression for *head loss* on a distance *L* according *Fig. 8.17*? What is the *head loss equation for steady uniform flow*?
16. Write down the *Darcy's equation foropen channel flow* . What every member in the formula presents?
17. Write down the *Chezy formula for the average velocity in open channel flow*! What every member in the formula presents?
18. Give the definitions for *drag force* and *lift force* and write down the corresponding equations! What are the general dependence expressions for the *drag* and *lift coefficients*?

Fluid Mechanics

Theory Homework No. 8

1. Give the definition for *free turbulence*!
2. Give the definition for *diffusion*! For which flows the diffusion is characteristic?
3. What are the definitions for *turbulent jets*, *buoyant jets* and *plumes* (*Fig. 8.23*)?
4. Define the term of *entrainment*! What is the result of the entrainment into a buoyant jet?
5. Give a definition for *dispersion of air pollution*! Which fluid motions are characteristic for this dispersion?
6. What are the bases for *atmospheric dispersion modelling*?
7. Give the definition for *multiphase flow*! Give a general quotation of the processes and equations which govern this flow!
8. Define *two-phase flow* in fluid mechanics! Give characteristic examples for two-phase flow!
9. Which *features* make *two-phase flow* an interesting and challenging branch of fluid mechanics?
10. Why it is important the modelling of gas-solid multiphase flow? For which processes of the process plants operation, the prediction of gas-solid flow fields is crucial?